Wavelets and Signal Processing

Reinhold Schneider

Sommersemester 2000

Recommended Literature

[1] Stéphane Mallat [1998]: *A Wavelet Tour of Signal Processing*, Academic Press, Inc.

[2] Ingrid Daubechies [1992]: *Ten Lectures on Wavelets*, SIAM, Philadelphia

[3] Charles K. Chui [1992]: *An Introduction to Wavelets* in Wavelet Analyis and its Applications, Vol. 1, Academic Press, Inc.

[4] A. V. Oppenheim and R. W. Schafer [1989]: *Discrete-Time Signal Processing*, Prentice-Hall, Englewood Cliffs

Contents

1 Fourier Analysis

With the observation (1807) that a continuous periodical function can be decomposed in a series of trigonometrical functions Fourier has put the foundation stone for one of the most important tools of modern mathematics - the Fourier analysis.

Because of its deep practical as well as theoretical impact and fundamental character the most essential results shall be summarized in this chapter.

1.1 Introduction

Definition 1.1 *Let $f : \mathbb{R} \to \mathbb{C}$ where $f \in L^1(\mathbb{R})$, i.e. $\int_{-\infty}^{\infty} |f(t)|\ \mathrm{d}t < \infty$, and $\xi \in \mathbb{R}$. Then the* **Fourier transform of f at point ξ** *is defined by the (Lebesgue) integral*

$$\boxed{\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} f(t)\ \mathrm{d}t.} \tag{1.1}$$

The function $\xi \to \hat{f}(\xi)$ is called the **Fourier transform of f**.

Note that the Fourier transform is well defined, i.e. the integral always exists since

$$\left| \int_{-\infty}^{\infty} e^{-i\xi t} f(t)\ \mathrm{d}t \right| \leq \int_{-\infty}^{\infty} |f(t)|\ \mathrm{d}t = \|f\|_{L_1}. \tag{1.2}$$

The auditors who are familiar with measure theory and the Lebesgue measure will observe that in most of the following proofs details related with measure theory are ommitted in favour of a compact and simplified representation. You can regard the completion of the proofs as an exercise for you.

Lemma 1.2 'Riemann-Lebesgue'
Let $f \in L^1(\mathbb{R})$. Then its Fourier transform $\hat{f}$ is uniformly continuous on $\mathbb{R}$. Furthermore $\hat{f}(\xi) \to 0$, as $\xi \to \infty$ or $\xi \to -\infty$.

Proof: We already know from (1.2) that $\hat{f} \in L^\infty(\mathbb{R})$ with $\|\hat{f}\|_\infty \leq \|f\|_1$.

Let $|\omega - \xi| < \delta$, then

$$|\hat{f}(\xi) - \hat{f}(\omega)| \leq \int_{-\infty}^{\infty} |f(t)||e^{-i\xi t} - e^{-i\omega t}|\ \mathrm{d}t.$$

Now, since $|e^{-i\xi t} - e^{-i\omega t}||f(x)| \leq 2|f(x)| \in L^1(\mathbb{R})$ and $|e^{-i\xi t} - e^{-i\omega t}| \to 0$ as $\delta \to 0$, the Lebesgue dominated convergence theorem implies that the quantity above tends to zero as $\delta \to 0$.

To prove the second part, we first assume that f' exists and is in $L^1(\mathbb{R})$. Integrating (1.1) by parts yields

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} f(t)\ \mathrm{d}t = -\frac{1}{i\xi}\left[e^{-i\xi t} f(t)\right]_{-\infty}^{\infty} + \frac{1}{\xi i}\int_{-\infty}^{\infty} e^{-i\xi t} f'(t)\ \mathrm{d}t.$$

From the continuity of $f \in L^1(\mathbb{R})$ we know $f(t) \to 0$ as $t \to \pm\infty$ and hence $\widehat{f'(\xi)} = i\xi\hat{f}(\xi)$:

$$|\hat{f}(\xi)| = \frac{1}{|\xi|}|\hat{f}'| \leq \frac{1}{|\xi|}\|f\|_1 \to 0, \quad \text{as} \quad \xi \to \pm\infty.$$

In general, for any given $\varepsilon > 0$ we can find a function g such that $g, g' \in L^1(\mathbb{R})$ and $\|f - g\|_1 < \varepsilon$. Thus, we have

$$\begin{aligned} |\hat{f}(\xi)| &\leq |\hat{f}(\xi) - \hat{g}(\xi)| + |\hat{g}(\xi)| \\ &\leq \int_{-\infty}^{\infty} |f(t) - g(t)| \left|e^{-i\xi t}\right| \, \mathrm{d}t + |\hat{g}(\xi)| \\ &= \|f - g\|_1 + |\hat{g}(\xi)| < \varepsilon + |\hat{g}(\xi)| \end{aligned}$$

completing the proof. □

The following example is of fundamental importance.

Lemma 1.3 *The Fourier transform of* $t \to f(t) = e^{-t^2}$ *(Gaussian function) is*

$$\hat{f}(\xi) = \sqrt{\pi} e^{-\frac{\xi^2}{4}}.$$

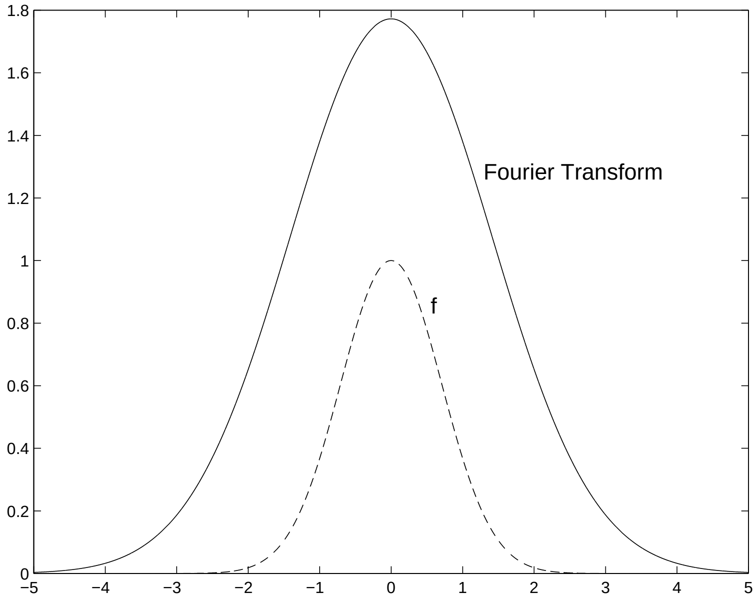


Figure 1.1: The Gaussian function and its Fourier transform

Proof: Consider the function

$$g(y) := \int_{-\infty}^{\infty} e^{-x^2+xy} \, \mathrm{d}x$$

By completing squares and setting $u := x - \frac{y}{2}$ we have

$$g(y) = \int_{-\infty}^{\infty} e^{-(x-\frac{y}{2})^2+\frac{y^2}{4}} \, \mathrm{d}x = e^{\frac{y^2}{4}} \int_{-\infty}^{\infty} e^{-u^2} \, \mathrm{d}u = \sqrt{\pi} e^{\frac{y^2}{4}}. \tag{1.3}$$

Now ,since $h(y) := \sqrt{\pi} e^{\frac{y^2}{4}}$ and $g(y)$ can be extended to be entire (analytic) functions, and since they agree on $\mathbb{R}$ as shown above, they must agree on the whole complex plane $\mathbb{C}$. In particular, by setting y to be $-i\omega$ the equality (1.3) becomes

$$g(-i\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} e^{-t^2} \, \mathrm{d}t = \sqrt{\pi} e^{-\frac{\xi^2}{4}}.$$

□

Definition 1.4 *Let $f, g \in L^1(\mathbb{R})$ then*

$$\boxed{h(x) = (f * g)(x) := \int_{-\infty}^{\infty} f(x-t) g(t) \, \mathrm{d}t}$$

is the **convolution** *of f and g.*

Convolution is an important composition in signal analysis. For example the quite popular linear time-invariant filters are completely characterized by a convolution with a so called transfer function. The following theorem is one of the most fundamental in signal processing and serves as the basis for the construction, analysis and fast implementation of digital filters.

Theorem 1.5 *Let $f, g \in L^1(\mathbb{R})$ then $h = (f * g) \in L^1(\mathbb{R})$ and furthermore the formula*

$$\boxed{\hat{h}(\xi) = (\widehat{f * g})(\xi) = \hat{f}(\xi) \hat{g}(\xi)} \tag{1.4}$$

holds for almost every $\xi \in \mathbb{R}$.

Proof: Applying Fubinis theorem yields

$$\begin{aligned} \|h\|_1 = \int_{-\infty}^{\infty} |h(x)| \, \mathrm{d}x &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-t) g(t) \, \mathrm{d}t \right| \, \mathrm{d}x \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-t)||g(t)| \, \mathrm{d}t \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(t)||f(x-t)| \, \mathrm{d}x \, \mathrm{d}t \\ (u := x-t) \ \Rightarrow \quad &= \left(\int_{-\infty}^{\infty} |g(t)| \, \mathrm{d}t \right) \left(\int_{-\infty}^{\infty} |f(u)| \, \mathrm{d}u \right) \\ &= \|g\|_1 \|f\|_1. \end{aligned}$$

Since $|f(x-t)||g(t)| \in L^1(\mathbb{R} \times \mathbb{R})$ we may consider

$$\begin{aligned} \hat{h}(\xi) &= \int_{-\infty}^{\infty} e^{-ix\xi} \int_{-\infty}^{\infty} f(x-t) g(t) \, \mathrm{d}t \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix\xi} f(x-t) g(t) \, \mathrm{d}t \, \mathrm{d}x. \end{aligned}$$

Again, by applying Fubinis theorem and substituting $u := x - t$ we obtain

$$\begin{aligned}\hat{h}(\xi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\xi(t+u)} f(u) g(t)\ \mathrm{d}u\ \mathrm{d}t \\ &= \left(\int_{-\infty}^{\infty} e^{-i\xi u} f(u)\ \mathrm{d}u\right)\left(\int_{-\infty}^{\infty} e^{-i\xi t} g(t)\ \mathrm{d}t\right)\end{aligned}$$

which proves (1.4). □

Theorem 1.6 Inverse Fourier transform

Let $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$. Then

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi t}\ \mathrm{d}\xi} \tag{1.5}$$

at each point t on the real axis where f is continuous.

Proof: Let us consider the convolution $(f * g_\alpha)(t)$ with the (more general) Gaussian function

$$g_\alpha(t) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{t^2}{4\alpha}}.$$

For those who are familiar with statistics; g_α is the probability density function of a normal distribution with zero mean and variance $\sqrt{2\alpha}$. First, we will show that as α tends to 0^+ the convolution term will converge to $f(t)$ at each point t where f is continuous. For an arbitrary $\varepsilon > 0$ we select an $\eta > 0$ such that

$$|f(t-\tau) - f(t)| < \varepsilon$$

for all $\tau \in \mathbb{R}$ with $|\tau| < \eta$. Then

$$\begin{aligned}|(f * g_\alpha)(t) - f(t)| &= \left|\int_{-\infty}^{\infty} f(t-\tau) g_\alpha\ \mathrm{d}\tau - f(t)\overbrace{\int_{-\infty}^{\infty} g_\alpha(\tau)\ \mathrm{d}\tau}^{=1}\right| \\ &= \left|\int_{-\infty}^{\infty} [f(t-\tau) - f(t)] g_\alpha(\tau)\ \mathrm{d}\tau\right| \\ &\le \int_{-\eta}^{\eta} |f(t-\tau) - f(t)| g_\alpha(\tau)\ \mathrm{d}\tau + \int_{|\tau|\ge\eta} |f(t-\tau) - f(t)| g_\alpha(\tau)\ \mathrm{d}\tau \\ &\le \varepsilon\int_{-\eta}^{\eta} g_\alpha(\tau)\ \mathrm{d}\tau + \|f\|_1 \max_{|\tau|\ge\eta} g_\alpha(\tau) + |f(t)|\int_{|\tau|\ge\eta} g_\alpha(\tau)\ \mathrm{d}\tau \\ &\le \varepsilon\int_{-\infty}^{\infty} g_\alpha(\tau)\ \mathrm{d}\tau + \|f\|_1 g_\alpha(\eta) + |f(t)|\int_{|\tau|\ge\eta/\sqrt{\alpha}} g_1(\tau)\ \mathrm{d}\tau \\ &= \varepsilon + \|f\|_1 g_\alpha(\eta) + |f(t)|\int_{|\tau|\ge\eta/\sqrt{\alpha}} g_1(\tau)\ \mathrm{d}\tau\end{aligned}$$

Since both $g_\alpha(\eta)$ and the last term converge to zero as α tends to 0^+, this completes the first part of the proof, $f * g_\alpha \to f$ for $\alpha \to 0^+$.

In the second part we will show that $(f * g_\alpha)(t)$ converges to the right hand side of equation (1.5). Consider the function

$$h(x) := \frac{1}{2\pi} e^{ixt} e^{-\alpha x^2}.$$

We compute its Fourier transform using the same technique as in the proof of Lemma 1.3

$$\begin{aligned} \hat{h}(\xi) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} e^{ixt} e^{-\alpha x^2} \, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(\xi - t)x} e^{-\alpha x^2} \, \mathrm{d}x \\ &= \frac{1}{2\pi} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{(\xi-t)^2}{4\alpha}} = g_\alpha(t - \xi). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} (f * g_\alpha)(t) &= \int_{-\infty}^{\infty} f(\xi) g_\alpha(t - \xi) \, \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} f(\xi) \hat{h}(\xi) \, \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \hat{f}(\omega) h(\omega) \, \mathrm{d}\omega \quad \text{(by applying Fubinis theorem)} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) e^{-\alpha\omega^2} \, \mathrm{d}\omega. \end{aligned}$$

Since $e^{-\alpha\omega^2}$ tends to 1 as $\alpha \to 0^+$, this completes the proof. □

The definition of the Fourier transform is not best suited for the function space $L^1(\mathbb{R})$. Note that in the assumptions of Theorem 1.6 both, f and $\hat{f}$, have to be absolutely integrable. Although for the proper definition of $\hat{f}$ the function f should be in $L^1(\mathbb{R})$ there is no good reason for $\hat{f}$.

Example 1.7 Let

$$\chi(t) = \begin{cases} 1 & \text{for } t \in [-1, 1] \\ 0 & \text{else} \end{cases}$$

be the characteristic function on the interval $[-1, 1]$. The associated Fourier transform is

$$\hat{\chi}(\xi) = \int_{-1}^{1} e^{-it\xi} \, \mathrm{d}t = \frac{1}{i\xi}(e^{i\xi} - e^{-i\xi}) = 2\frac{\sin\xi}{\xi}$$

which is a wildly oscillating function and not absolutely integrable. However, the integral $\int_{-\infty}^{\infty} |\hat{\chi}(\xi)|^2 \, \mathrm{d}\xi$ is finite.

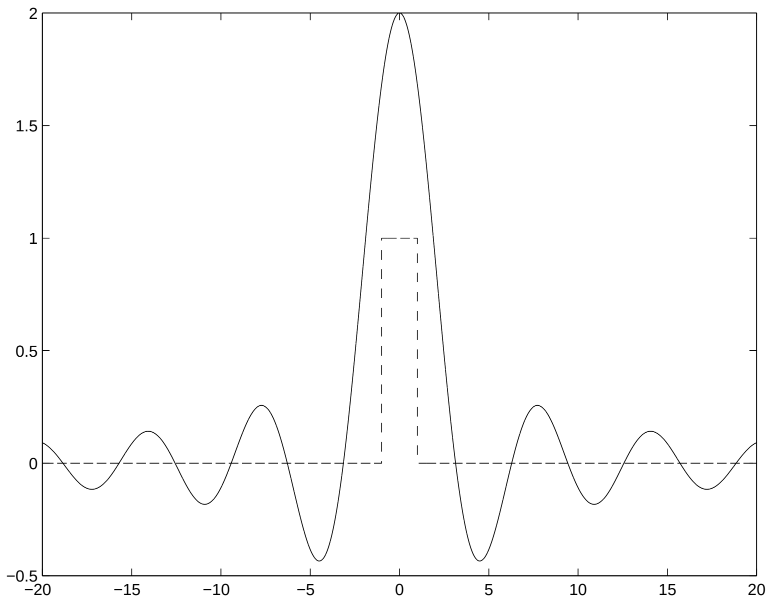


Figure 1.2: The characteristic function and its Fourier transform

Motivated by the previous example we consider a more convenient function space:

$$L^2(\mathbb{R}) := \left\{ u : \ \|u\|_2 := \left(\int_{-\infty}^{\infty} |u(t)|^2 \ \mathrm{d}t \right)^{\frac{1}{2}} < \infty \right\}$$

$L^2(\mathbb{R})$ is a so called **Hilbert space** with the "inner product" (scalar product)

$$\langle u, v \rangle := \int_{-\infty}^{\infty} u(x)\overline{v(x)} \ \mathrm{d}x$$

which induces an associated norm $\|u\|_2 = \sqrt{\langle u, u \rangle}$.

Theorem 1.8 Parseval and Plancherel

Let $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the following holds

$$\boxed{\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)} \ \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)} \ \mathrm{d}\xi = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle,} \tag{1.6}$$

and hence

$$\boxed{\|f\|_2^2 = \langle f, f \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{f} \rangle = \frac{1}{2\pi} \|\hat{f}\|_2^2.} \tag{1.7}$$

Proof: Let $\tilde{g}(t) = \overline{g(-t)}$ and $h(t) = (f * \tilde{g})(t)$, then the application of the convolution Theorem 1.5 yields

$$\hat{h}(\xi) = \hat{f}(\xi)\hat{\tilde{g}}(\xi) = \hat{f}(\xi)\overline{\hat{g}(\xi)}.$$

Using Theorem 1.6 about the inverse Fourier transform we get

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\overline{g(t)}\,\mathrm{d}t &= \int_{-\infty}^{\infty} f(-t)\overline{g(-t)}\,\mathrm{d}t \\ = h(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}(\xi)\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi.\end{aligned}$$

□

As a consequence of this theorem, we observe that $\mathcal{F} : f \to \hat{f}$ can be considered as a bounded linear operator on $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ with range in $L^2(\mathbb{R})$. Equation (1.7) tells us that $\|\mathcal{F}\| = \sqrt{2\pi}$. Since $L^2(\mathbb{R})$ is dense in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $\mathcal{F}$ has a normpreserving extension to all elements of $L^2(\mathbb{R})$. Furthermore the equations (1.6) and (1.7) hold for all $f, g \in L^2(\mathbb{R})$ so that the domain of $\mathcal{F}$ can be extended to the whole $L^2(\mathbb{R})$.

The following theorem shows the beautiful connection between $\mathcal{F}$ and $L^2(\mathbb{R})$.

Theorem 1.9 *The Fourier transform $\mathcal{F}$ is a linear one-to-one map of $L^2(\mathbb{R})$ onto itself. In other words, to every $g \in L^2(\mathbb{R})$, there corresponds one and only one $f \in L^2(\mathbb{R})$ such that $\hat{f} = g$.*

Proof: For the proof and related details see [3]. □

A major drawback of the Fourier analysis is that f and $\hat{f}$ cannot simultaneously be localized on the corresponding domains.

Theorem 1.10 *If $f \not\equiv 0$ has compact support then $\hat{f}(\omega)$ cannot vanish on any interval $[a, b] \subset \mathbb{R}$.*

Proof: Let f be zero outside the interval $[-M, M]$ then

$$\hat{f}(\omega) = \int_{-M}^{M} f(t)e^{-i\omega t}\,\mathrm{d}t.$$

Assume $\hat{f}(\omega)$ to be zero for all $\omega \in [c, d]$ then also all its derivatives vanish on the interior of this interval. Let us consider a $w_0 \in (c, d)$ and differentiate n-times with respect to ω

$$0 = \hat{f}^{(n)}(\omega_0) = \int_{-M}^{M} f(t)(-it)^n e^{-i\omega_0 t}\,\mathrm{d}t.$$

Using the equality

$$\hat{f}(\omega) = \int_{-M}^{M} f(t)e^{-it(\omega-\omega_0)}e^{-it\omega_0}\,\mathrm{d}t.$$

and from the power series expansion of

$$e^{-it(\omega-\omega_0)} = \sum_{k=0}^{\infty} \frac{[-it(\omega-\omega_0)]^k}{k!}$$

we get

$$\hat{f}(\omega) = \sum_{k=0}^{\infty} \frac{(\omega-\omega_0)^k}{k!}\int_{-M}^{M} f(t)(-it)^k e^{-it\omega_0}\,\mathrm{d}t = 0.$$

Hence $\hat{f} \equiv 0 \;\Rightarrow\; f \equiv 0$ which contradicts the assumption. □

Summary 1.11 Properties of the Fourier transform

Property	**Function**	**Fourier transform**
	$t \to f(t)$	$\xi \to \hat{f}(\xi)$
linearity	$\alpha f(t) + \beta g(t)$	$\alpha \hat{f}(\xi) + \beta \hat{g}(\xi)$
convolution	$(f * g)(t)$	$\hat{f}(\xi)\hat{g}(\xi)$
translation	$f(t - \tau)$	$e^{-i\tau\xi}\hat{f}(\xi)$
scaling	$f(\frac{t}{s})$	$\|s\|\hat{f}(s\xi)$
derivative	$f'(t)$	$i\xi\hat{f}(\xi)$
inverse	$\hat{f}(t)$	$2\pi f^{-}(\xi)$
reflection 1	$\overline{f^{-}(t)}$	$\overline{\hat{f}(\xi)}$
reflection 2	$f^{-}(t)$	$(\hat{f})^{-}(\xi)$
Gaussian function	e^{-t^2}	$\sqrt{\pi}e^{-\frac{\xi^2}{4}}$
characteristic function	$\chi_{[-1,1]}$	$2\frac{\sin\xi}{\xi}$
Dirac distribution	$\delta_x(t)$	$e^{-i\xi x}$
	$\frac{1}{1+x^2}$	$\pi e^{-\|\xi\|}$

The Dirac distribution δ_x is a functional $C^\infty \to \mathbb{R}$ defined via $\delta_x(\varphi) = \varphi(x)$, $\forall \varphi \in C_0^\infty(\mathbb{R})$. The function f^- is called the **reflection** of f and is defined as $f^-(t) := f(-t)$.

1.2 Uncertainty Principle

Let us define the **variances** of f in the time and the frequency domains by

$$\sigma_t^2(x) = \frac{1}{\|f\|_2^2} \int_{-\infty}^{\infty} (t - x)^2 |f(t)|^2 \, \mathrm{d}t$$

and

$$\sigma_\omega^2(\xi) = \frac{1}{2\pi\|f\|_2^2} \int_{-\infty}^{\infty} (\omega - \xi)^2 |\hat{f}(\omega)|^2 \, \mathrm{d}\omega.$$

Then we have the following result:

Proposition 1.12 Heisenberg Uncertainty

a) $\sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4}$,

b) $\sigma_t^2 \sigma_\omega^2 = \frac{1}{4}$ *iff there exists* $(x, \xi, a, z) \in \mathbb{R}^2 \times \mathbb{C}^2$ *such that* $\hat{f}(t) = ae^{i\xi t}e^{-z(t-x)^2}$.

Proof: The following proof requires the assumption that $\sqrt{t}f(t) \to 0$ for $t \to \infty$. Indeed the result is valid for $f \in L^2(\mathbb{R})$. Obviously for $x = \xi = 0$ we have

$$\sigma_t^2(0)\sigma_\omega^2(0) = \frac{1}{2\pi\|f\|^4}\int_{-\infty}^{\infty} t^2|f(t)|^2 \,\mathrm{d}t \cdot \int_{-\infty}^{\infty} \omega^2|\hat{f}(\omega)|^2 \,\mathrm{d}\omega,$$

and obtain from Plancherels formula and the Cauchy-Schwarz inequality

$$\begin{aligned} \sigma_t(0)^2\sigma_\omega(0)^2 &= \frac{1}{\|f\|^4}\int_{-\infty}^{\infty} |tf(t)|^2 \,\mathrm{d}t \cdot \int_{-\infty}^{\infty} |f'(t)|^2 \,\mathrm{d}t \\ &\geq \frac{1}{\|f\|^4}\left(\int_{-\infty}^{\infty} |tf'(t)\overline{f(t)}| \,\mathrm{d}t\right)^2 \\ &\geq \frac{1}{\|f\|^4}\left(\int_{-\infty}^{\infty} \frac{t}{2}|f'(t)\overline{f(t)} + \overline{f'(t)}f(t)| \,\mathrm{d}t\right)^2 \\ &= \frac{1}{4\|f\|^4}\left(\int_{-\infty}^{\infty} t(|f(t)|^2)' \,\mathrm{d}t\right)^2. \end{aligned}$$

Integration by parts together with the assumption $\lim_{t\to\infty} \sqrt{t}f(t) = 0$ yields

$$\sigma_t(0)^2\sigma_\omega(0)^2 \geq \frac{1}{4\|f\|^4}\left(\int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t\right)^2 = \frac{1}{4}.$$

For $x \neq 0$ or $\xi \neq 0$ we substitute

$$z := (t - x) \quad \text{and} \quad \eta := (\omega - \xi)$$

and by the same arguments we receive the same result.

The proof of part b) is left to the diligent reader. □

This result is somehow disappointing since it guarantees that it is never possible to localize the energy spreads of f in the time and frequency domains arbitrarily exact.

1.3 Linear Time-Invariant Operators (Filtering)

Definition 1.13 *A linear operator $L : C_0^\infty(\mathbb{R}) \to C(\mathbb{R})$ (resp. $L^p(\mathbb{R})$) is called* **time-invariant** *if it commutes with the time translation operator T_τ defined by $T_\tau(f(\cdot)) = f(\cdot - \tau)$, i.e.,*

$$T_\tau Lf(t) = (Lf)(t - \tau) = LT_\tau f(t) = (Lf(\cdot - \tau))(t).$$

We denote the Dirac distribution located at $\tau \in \mathbb{R}$ by δ_τ where

$$\delta_\tau(\varphi) = \int_{-\infty}^{\infty} \delta(t - \tau)\varphi(t) \,\mathrm{d}t = \varphi(\tau)$$

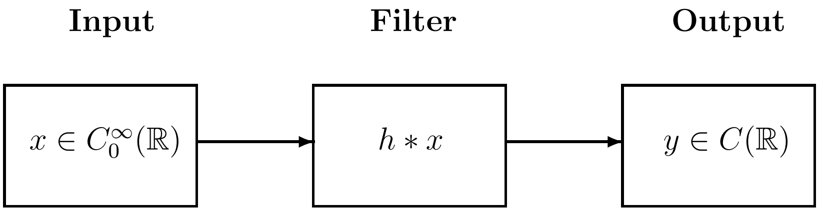
and write for the sake of simplicity

$$\delta_\tau = \delta(\cdot - \tau) \quad \text{and} \quad \varphi(\tau) = \langle \delta(\cdot - \tau), \varphi \rangle.$$

Let $L^\star$ be the dual operator of L and $h = L^\star \delta_0$, then

$$\begin{aligned}(L\varphi)(\tau) &= \langle \delta(\cdot - \tau), L\varphi\rangle = \langle \delta_0, (L\varphi)(\cdot + \tau)\rangle \\ &= \langle \delta_0, (L\varphi(\cdot + \tau))(\cdot)\rangle = \langle L^\star \delta_0, \varphi(\cdot + \tau)\rangle \\ &= \langle (L^\star \delta_0)(\cdot - \tau), \varphi\rangle = \langle h(\cdot - \tau), \varphi\rangle \\ &= (h * \varphi)(\tau).\end{aligned}$$

Thus, any linear time-invariant operation may be rewritten as a convolution which is a remarkable property of LTI operators.



The function h is called the **transfer function** or **impulse response** of L. The following equalities hold:

1. $h * \varphi = \varphi * h$,
2. $\frac{\mathrm{d}}{\mathrm{d}t}(h * \varphi)(t) = \left(\frac{\mathrm{dh}}{\mathrm{d}t} * \varphi\right)(t) = \left(h * \frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)(t)$.

Definition 1.14 *A convolution with $h = L^\star \delta_0$ is called a* **linear time-invariant filtering**. *Furthermore a filter h is called* **causal** *if supp $h \subseteq \mathbb{R}_+$ and* **stable** *if $h \in L^1(\mathbb{R})$.*

Stability implies for every $\varphi \in C^0(\mathbb{R})$:

$$\begin{aligned}|(h * \varphi)(t)| &= \left|\int_{-\infty}^{\infty} h(t - \tau)\varphi(\tau)\ \mathrm{d}\tau\right| = \left|\int_{-\infty}^{\infty} \varphi(t - \tau)h(\tau)\ \mathrm{d}\tau\right| \\ &\leq \sup_{t \in \mathbb{R}} |\varphi(t)| \int_{-\infty}^{\infty} |h(\tau)|\ \mathrm{d}\tau = \|\varphi\|_{C^0} \cdot \|h\|_{L^1}. \\ &\Rightarrow \|h * \varphi\|_{L^\infty} \leq \|\varphi\|_{C^0} \cdot \|h\|_{L^1}\end{aligned}$$

In other words, stable filters do not amplify the input data. Causality is nothing but the independence of the output data on input data from the future.

Example 1.15 Time averaging over an interval of length T.
As a trivial example for LTI filter design consider

$$h(t) = \frac{1}{T}\chi_{[-T/2,T/2]} \quad \Rightarrow \quad (h * \varphi)(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} \varphi(\tau)\ \mathrm{d}\tau.$$

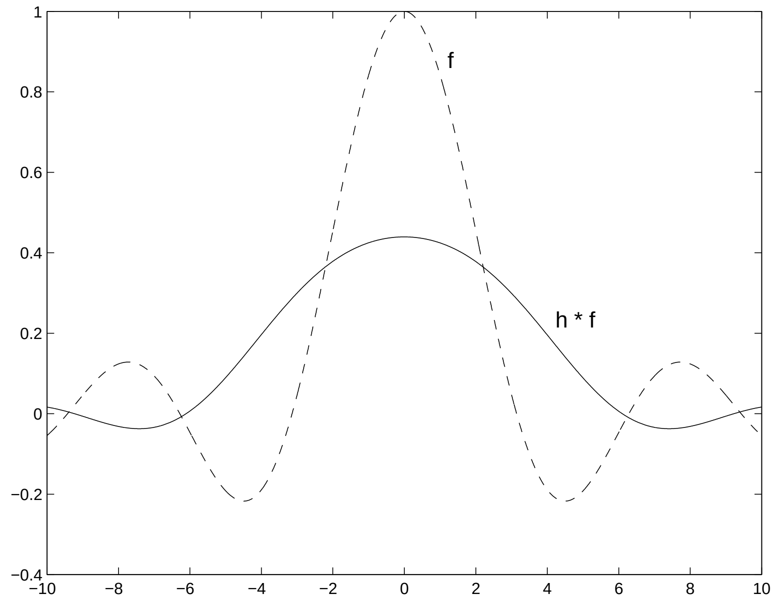


Figure 1.3: Time averaging for $f(t) = \frac{\sin t}{t}$ and $T = 8$

Example 1.16 Gibbs oscillation

Let us consider an ideal low pass filter with band width 2ξ described by the Fourier transform of the transfer function,

$$\hat{h}_\xi(\omega) = \begin{cases} 1, & \omega \in [-\xi, \xi] \\ 0, & \text{otherwise.} \end{cases}$$

This is a more useful application of LTI filters, it is often desirable in sound processing to remove all frequencies which are greater than a certain treshold from the input signal. However, the obtained output data is not necessarily the sound you would like to have in the back of your car. The following result demonstrates that (Gibbs) oscillations are created by $f * h_\xi$:

For any $\xi > 0$ and the staircase function

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

one has

$$(u * h_\xi)(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\xi t} \frac{\sin \tau}{\tau} \, \mathrm{d}\tau. \tag{1.8}$$

Proof: Application of Theorem 1.6 about the inverse Fourier transform yields

$$\begin{aligned} h_\xi(t) &= \frac{1}{2\pi} \int_{-\xi}^{\xi} e^{it\omega} \, \mathrm{d}\omega = \frac{1}{2\pi} \frac{1}{it} e^{it\omega} \Big|_{\omega=-\xi}^{\xi} \\ &= \frac{1}{\pi} \frac{\sin \xi t}{t}, \end{aligned}$$

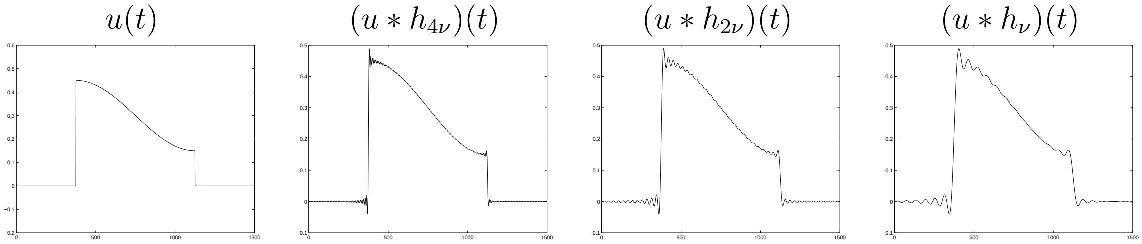


Figure 1.4: Gibbs oscillations created by ideal low-pass filters with cut-off frequencies that decrease from left to right.

and hence

$$(u * h_\xi)(t) = \frac{1}{\pi}\int_0^\infty \frac{\sin\,\xi(t-\tau)}{t-\tau}\,\mathrm{d}\tau \overset{x:=\xi(t-\tau)}{=} \frac{1}{\pi}\int_{-\infty}^{\xi t}\frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\sin x}{x}\,\mathrm{d}x + \frac{1}{\pi}\int_0^{\xi t}\frac{\sin x}{x}\,\mathrm{d}x$$

which proves (1.8) . □

Example 1.17 Passive Electronic Circuit
Let the input voltage $f(t)$ and output voltage $g(t)$ satisfy the ordinary differential equation

$$\sum_{k=0}^{K} a_k f^{(k)}(t) = \sum_{k=0}^{M} b_k g^{(k)}(t). \tag{1.9}$$

Then

$$\sum_{k=0}^{K} a_k (i\xi)^k \hat{f}(\xi) = \sum_{k=0}^{M} b_k (i\xi)^k \hat{g}(\xi).$$

which gives

$$\hat{g}(\xi) = \frac{\sum_{k=0}^{K} a_k (i\xi)^k}{\sum_{k=0}^{M} b_k (i\xi)^k}\hat{f}(\xi).$$

Therefore the impulse response of (1.9) satisfies

$$\hat{h}(\xi) = \frac{\sum_{k=0}^{K} a_k (i\xi)^k}{\sum_{k=0}^{M} b_k (i\xi)^k}.$$

The roots of the polynomials $\sum_{k=0}^{K} a_k (i\xi)^k$ and $\sum_{k=0}^{K} b_k (i\xi)^k$ are called **zeros** and **poles** of the linear system (1.9) which are significant values in linear filter design. For more details see e.g. [4].

2 Discrete-Time Signal Processing

Digital signal processing plays an important role in many areas like speech processing, television, tape recording and all other types of information manipulation. Whether sound recordings or images, most discrete signals are obtained by sampling an analog signal so that we can expect some parallelism to section 1.3. In this section we will study conditions for reconstructing an analog signal from a uniform sampling. Once more, the Fourier transformation is unavoidable because the eigenvectors of discrete linear time-invariant operators are sinusoidal waves. The Fourier transform may be discretized for signals of finite size and implemented with fast computational algorithms.

2.1 Sampling

Let us consider the 'Dirac comb', for $T > 0$

$$\mathcal{C} := \sum_{k=-\infty}^{\infty} \delta_{kT}$$

which is defined by

$$\mathcal{C}(\varphi) = \langle \mathcal{C}, \varphi \rangle = \sum_{k=-\infty}^{\infty} \varphi(kT), \quad \forall \varphi \in C_0^\infty(\mathbb{R}).$$

We write $\delta_{kT} := \delta(t-kT)$ keeping in mind that $\delta(t-kT)$ is *not* a function value at $t-kT$. The Fourier transform of $\mathcal{C}$ is defined by

$$\hat{\mathcal{C}}(\varphi) := \mathcal{C}(\hat{\varphi}) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(kT)$$

which is equal to

$$\hat{\mathcal{C}}(\varphi) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ikTt} \varphi(t) \, \mathrm{d}t = \Big\langle \sum_{k=-\infty}^{\infty} e^{-ikT\cdot}, \varphi(\cdot) \Big\rangle.$$

Therefore we can write

$$\hat{\mathcal{C}}(\xi) = \sum_{k=-\infty}^{\infty} e^{-ikT\xi}, \quad \xi \in \mathbb{R}.$$

Theorem 2.1 Poisson-Summation Formulas

In the sense of distributions the following two equations hold for $\varphi \in C_0^\infty(\mathbb{R})$:

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \hat{\varphi}(kT) &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ikT\xi} \varphi(\xi) \, \mathrm{d}\xi = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{-ikT\xi} \varphi(\xi) \, \mathrm{d}\xi \\ &= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \varphi\left(\frac{2\pi k}{T}\right), \end{aligned} \tag{2.1}$$

and

$$\boxed{\sum_{k=-\infty}^{\infty} e^{-ikT\xi} = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta_{\frac{2\pi k}{T}}(\xi).} \tag{2.2}$$

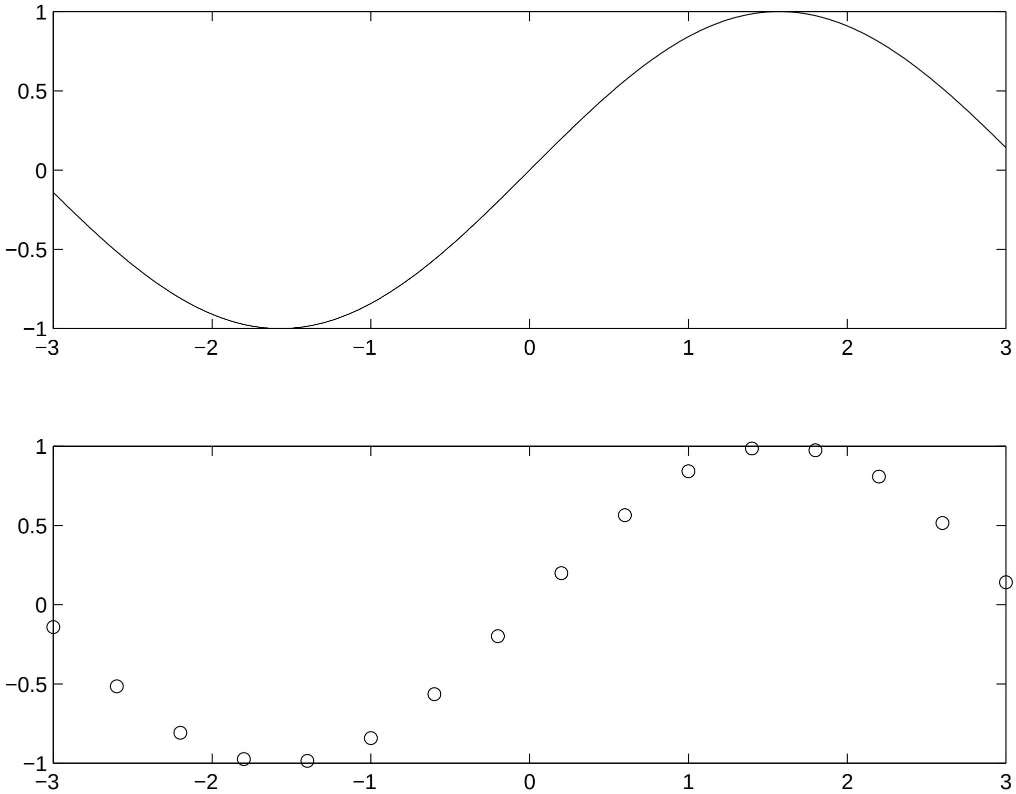


Figure 2.1: Sampling $f(t) = \sin t$ with the Dirac comb.

Proof: It is sufficient to consider $\varphi \in C_0^\infty(\mathbb{R})$ with support in the interval $\varphi \subseteq \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$. In this case the Poisson formula (2.1) reduces to

$$\lim_{N\to\infty} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \sum_{k=-N}^{N} e^{-ikT\xi}\varphi(\xi)\ \mathrm{d}\xi = \frac{2\pi}{T}\varphi(0). \tag{2.3}$$

The geometric series gives for $\xi \neq 0$:

$$\begin{aligned} \sum_{k=-N}^{N} e^{-ikT\xi} &= e^{-iT\xi N}\sum_{k=0}^{2N}(e^{-iT\xi})^k = \frac{e^{-i(N+1)T\xi} - e^{iNT\xi}}{e^{-iT\xi} - 1} \\ &= \frac{e^{-i(N+\frac{1}{2})T\xi} - e^{i(N+\frac{1}{2})T\xi}}{e^{-i\frac{T}{2}\xi} - e^{i\frac{T}{2}\xi}} = \frac{\sin\left(N+\frac{1}{2}\right)T\xi}{\sin\frac{T}{2}\xi}. \end{aligned}$$

Note that this formula is also consistent for the case $\xi = 0$, since

$$\lim_{\xi\to 0} \frac{\sin\left(N+\frac{1}{2}\right)T\xi}{\sin\frac{T}{2}\xi} = 2N+1 = \sum_{k=-N}^{N} 1.$$

We insert this result into the left hand side of (2.3) and obtain

$$\lim_{N\to\infty} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{\sin\left(N+\frac{1}{2}\right)T\xi}{T\xi} \cdot \frac{T\xi\ \varphi(\xi)}{\sin\frac{T}{2}\xi}\ \mathrm{d}\xi = \lim_{N\to\infty} \int_{-\infty}^{\infty} 2\frac{\sin\left(N+\frac{1}{2}\right)T\xi}{T\xi} \cdot \hat{\psi}(\xi)\ \mathrm{d}\xi$$

where

$$\hat{\psi}(\xi) = \begin{cases} \varphi(\xi)\frac{T\xi}{2\sin\frac{T}{2}\xi}, & \xi \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right] \\ 0, & \text{otherwise.} \end{cases}$$

The latter integral may be rewritten as

$$(2N+1)\left\langle \frac{\sin\left(N+\frac{1}{2}\right)T\cdot}{\left(N+\frac{1}{2}\right)T\cdot}, \hat{\psi} \right\rangle = \frac{\pi}{T}\left\langle \chi_{\left[-\left(N+\frac{1}{2}\right)T,\left(N+\frac{1}{2}\right)T\right]}, \psi \right\rangle$$

where we stressed Plancherels formula of Theorem 1.8 and a scaled version of Example 1.7. ψ denotes the Fourier preimage of $\hat{\psi}$. Altogether we get

$$\begin{aligned} \lim_{N\to\infty}\int_{-\infty}^{\infty} 2\frac{\sin\left(N+\frac{1}{2}\right)T\xi}{T\xi}\cdot\hat{\psi}(\xi)\,\mathrm{d}\xi &= \frac{2\pi}{T}\lim_{N\to\infty}\int_{-\left(N+\frac{1}{2}\right)T}^{\left(N+\frac{1}{2}\right)T}\psi(t)\,\mathrm{d}t \\ &= \frac{2\pi}{T}\hat{\psi}(0) = \frac{2\pi}{T}\varphi(0). \end{aligned}$$

□

A discrete signal can be represented as a sum of Dirac distributions. For any sample value $f(kT)$ at the sample point $t = kT,\ k \in \mathbb{Z},\ T > 0$, we associate the distribution $f(kT)\delta_{kT}$. A uniform sampling corresponds to the distribution

$$f_d = \sum_{k=-\infty}^{\infty} f(kT)\delta_{kT}.$$

The Fourier transform of f_d is represented by the Fourier series

$$\hat{f}_d(\xi) = \sum_{k=-\infty}^{\infty} f(kT)e^{-ikT\xi}. \tag{2.4}$$

Proposition 2.2 *The Fourier transform of the distribution f_d satisfies*

$$\hat{f}_d(\xi) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\hat{f}\left(\xi - \frac{2k\pi}{T}\right). \tag{2.5}$$

Proof: We have $f(kT)\delta_{kT}(\varphi) = f(t)\delta_{kT}(\varphi)$ and so that we can write

$$f_d(t) = f(t)\cdot\sum_{k=-\infty}^{\infty}\delta_{kT} = f(t)\cdot\mathcal{C}.$$

Computing the Fourier transform yields

$$\hat{f}_d(\xi) = \frac{1}{2\pi}(\hat{f} * \hat{\mathcal{C}})(\xi).$$

Inserting Poissons formula (2.2)

$$\hat{\mathcal{C}}(\xi) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta_{\frac{2\pi k}{T}}(\xi)$$

proves that

$$\frac{1}{2\pi}(\hat{f} * \hat{\mathcal{C}})(\xi) = \frac{1}{T} \int_{-\infty}^{\infty} \hat{f}(\xi - \omega) \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right) \, \mathrm{d}\omega = \frac{1}{T} \sum_{k=-\infty}^{\infty} \hat{f}\left(\xi - \frac{2\pi k}{T}\right).$$

□

The following result was primarily proved by Whittaker (1935) but rediscovered by Shannon (1949).

Theorem 2.3 Shannon, Whittaker
If supp $\hat{f} \subseteq \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$ *and* $h_T(t) = \frac{T}{\pi t} \sin \frac{\pi t}{T}$*, then*

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} f(kT) \cdot h_T(t - kT)} \tag{2.6}$$

holds.

Proof: From (2.5) in Proposition 2.2 we have

$$\hat{f}_d(\xi) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \hat{f}\left(\xi - \frac{2k\pi}{T}\right).$$

Since the frequency band is bounded, the supports of $\hat{f}(\cdot - \frac{2k\pi}{T}),\ k \in \mathbb{Z}$ have no overlap and thus

$$\hat{f}_d(\xi) = \frac{\hat{f}(\xi)}{T}, \quad \text{for } |\xi| \leq \frac{\pi}{T}.$$

The Fourier transform of h_T is $T\chi_{[-\pi/T, \pi/T]}$. Therefore $\hat{f}(\xi) = \hat{f}_d(\xi) \cdot \hat{h}_T(\xi)$ and by means of convolution Theorem 1.5,

$$f(t) = (h_T * f_d)(t) = \left(h_T * \sum_{k=-\infty}^{\infty} f(kT)\delta_{kT}\right)(t) = \sum_{k=-\infty}^{\infty} f(kT) \cdot h_T(t - kT).$$

□

Equality (2.6) allows us a decomposition of functions with bounded frequency band as a sum of sinusoidal waves.

2.2 Aliasing

The length of the sampling interval is restricted - by storage and efficiency requirements. Additionally the condition supp $\hat{f} \subseteq \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$ is not necessarily fulfilled. This causes an additional error which can be severe. Have you ever wondered why car tires are spinning backwards in ancient action movies ? Such effects are called **aliasing**.

Example 2.4 Let us consider a high frequency signal

$$f(t) = \cos \omega_0 t = \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})$$

whose Fourier transform is

$$\hat{f}(\xi) = \pi(\delta(\xi - \omega_0) + \delta(\xi + \omega_0)).$$

If $\frac{2\pi}{T} > \omega_0 > \frac{\pi}{T}$ we get

$$\begin{aligned} \hat{h}_T(\xi)\hat{f}_d(\xi) &= \pi\chi_{[-\frac{\pi}{T},\frac{\pi}{T}]} \sum_{k=-\infty}^{\infty} \left[\delta\left(\xi - \omega_0 - \frac{2k\pi}{T}\right) + \delta\left(\xi + \omega_0 - \frac{2k\pi}{T}\right)\right] \\ &= \pi\left[\delta\left(\xi - \omega_0 + \frac{2\pi}{T}\right) + \delta\left(\xi + \omega_0 - \frac{2\pi}{T}\right)\right]. \end{aligned}$$

Applying the inverse Fourier transform on both sides yields

$$(f_d * h_T)(t) = \cos\ t\left(\omega_0 - \frac{2\pi}{T}\right).$$

Thus, the *high* frequency signal is reduced to a *low* frequency signal with a frequency $\omega_0 - \frac{2\pi}{T} \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$.

As a more practical example you might try to compress a picture with sharp edges and lots of tiny details via JPEG.

The effect of aliasing can be minimized through approximating f by a suitable band limited signal f_b with supp $\hat{f}_b \subseteq \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$. The error can be measured using Theorem 1.8:

$$\begin{aligned} \|f - f_b\|^2_{L_2(\mathbb{R})} &= \frac{1}{2\pi}\|\hat{f} - \hat{f}_b\|^2_{L_2(\mathbb{R})} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi) - \hat{f}_b(\xi)|^2 \,\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} |\hat{f}(\xi) - \hat{f}_b(\xi)|^2 \,\mathrm{d}\xi + \int_{|\xi|>\pi/T} |\hat{f}(\xi) - \hat{f}_b(\xi)|^2 \,\mathrm{d}\xi. \end{aligned}$$

This is minimized iff $\hat{f}(\xi) = \hat{f}_b(\xi)$ for $|\xi| \leq \frac{\pi}{T}$. Setting

$$\hat{f}_b(\xi) = \chi_{[-\frac{\pi}{T},\frac{\pi}{T}]}\hat{f}(\xi)$$

is obviously a good way for doing this. Therefore sampling of the filtered signal $f_b(t) = (f * h_T)(t)$ will give best results.

In practice an analog to digital converter is composed of a low pass filter that limits to $\left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$ and uniform sampling at intervals T.

Proposition 2.5 *Let* $h_T(t) = \frac{T}{\pi t} \sin \frac{\pi t}{T}$*, then*

$$\{h_T(t - kT) \ : \ k \in \mathbb{Z}\} \tag{2.7}$$

forms an orthogonal family of functions in $U_T \subset L^2(\mathbb{R})$*.* U_T *is the space of all functions in* $L^2(\mathbb{R})$ *whose Fourier transform is supported in the interval* $\left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$*. Let* $f \in U_T$ *be continuous, then it may be represented as*

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT) \cdot h_T(t - kT) \tag{2.8}$$

and, conversely, the basis coefficients are given by

$$f(kT) = \frac{1}{T} \langle f, h(\cdot - kT) \rangle. \tag{2.9}$$

Proof: Remind that $\hat{h}_T = T\chi_{[-\frac{\pi}{T},\frac{\pi}{T}]}$. Parsevals formula (Theorem 1.8) yields

$$\begin{aligned}
\langle h_T(\cdot - kT), h_T(\cdot - nT) \rangle &= \frac{1}{2\pi} \langle e^{-ikT\cdot} \hat{h}_T(\cdot), e^{-inT\cdot} \hat{h}_T(\cdot) \rangle \\
&= \frac{T^2}{2\pi} \int_{-\infty}^{\infty} \chi_{[-\frac{\pi}{T},\frac{\pi}{T}]}(\xi) e^{-i(k-n)T\xi} \, \mathrm{d}\xi \\
&= \frac{T^2}{2\pi} \int_{-\pi/T}^{\pi/T} e^{-i(k-n)T\xi} \, \mathrm{d}\xi = \begin{cases} 0 & \text{for } k \neq n \\ T & \text{for } k = n \end{cases} .
\end{aligned}$$

Clearly each $h_T(\cdot - kT)$ is a member of U_T.

Shannons sampling Theorem 2.3 gives the first representation (2.8) and, again with Theorem 1.8, we compute

$$\begin{aligned}
\langle f, h_T(\cdot - kT) \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \hat{h}_T(\xi) e^{ikT\xi} \, \mathrm{d}\xi \\
&= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \hat{f}(\xi) e^{ikT\xi} \, \mathrm{d}\xi \\
(\text{since } f \in U_T) \ \Rightarrow \quad &= \frac{T}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ikT\xi} \, \mathrm{d}\xi = Tf(kT),
\end{aligned}$$

which proves (2.9). □

2.3 Discrete Time-Invariant Filters

To simplify notation, we now assume $T = 1$ and set

$$f[k] := f(kT), \quad k \in \mathbb{Z}$$

as the sample values. Now, $f : \mathbb{Z} \to \mathbb{C}$ where we can assume that $f \in l^2(\mathbb{Z})$. A **discrete linear time-invariant operator** L satisfies

$$(Lf[\cdot - n])[k] = Lf[k - n]. \tag{2.10}$$

The **discrete Dirac distribution** $\delta[k]$ is defined by $\delta[k] := \delta_k$. A discrete signal can be represented as

$$f[k] = \sum_{n=-\infty}^{\infty} f[n]\delta[k-n].$$

Any time-invariant linear operator L may be expressed by its action on the Dirac distributions. Let $L\delta[k] =: g[k]$, the so called **discrete impulse response**, then

$$Lf[k] = \sum_{n=-\infty}^{\infty} f[n]L\delta[k-n] = \sum_{n=-\infty}^{\infty} f[n]g[k-n].$$

If g has finite support then the above term reduces to a finite sum, such operators are called **finite impulse response Filters (FIR)**.

Since

$$|Lf[n]| \leq \sup_{k\in\mathbb{Z}} |f[k]| \cdot \sum_{k=-\infty}^{\infty} |g[k]|$$

we call L analogous to Definition 1.14 **stable** if $g \in l_1(\mathbb{Z})$.

Transfer functions

We consider

$$Le^{i\omega n} = \sum_{k=-\infty}^{\infty} e^{i\omega(n-k)}g[k] = e^{i\omega n}\sum_{k=-\infty}^{\infty} e^{-i\omega k}g[k] = \hat{g}(\omega)e^{i\omega n},$$

which shows that L has eigenvectors $e^{i\omega n}$ with eigenvalues $\hat{g}(\omega)$. The function $\hat{g}(\omega) = \sum_{k=-\infty}^{\infty} e^{-i\omega k}g[k]$ is called the **discrete filter transfer function**.

Example 2.6 Uniform discrete average

This example corresponds to the continuous version of Example 1.15.

$$Lf[n] = \frac{1}{2N+1}\sum_{k=n-N}^{N+n} f[k] = (g*f)[n],$$

where

$$g[k] = \frac{1}{2N+1}\begin{cases} 1 & k=-N,\ldots,N, \\ 0 & \text{otherwise,} \end{cases}$$

which is obviously a stable filter. The associated transfer function is

$$\hat{g}(\omega) = \frac{1}{2N+1}\sum_{k=-N}^{N} e^{-i\omega k} = \frac{1}{2N+1}\frac{\sin(N+\frac{1}{2})\omega}{\sin\frac{\omega}{2}}$$

(see also proof of Theorem 2.1).

2.4 Fourier Series

The Fourier transfrom of the discrete signal

$$f(t) = \sum_{k=-\infty}^{\infty} f[k]\delta(t-k)$$

is given by

$$\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-i\omega k}$$

where the latter one is a periodic function of period 2π. Therefore we consider the space $L^2[-\pi,\pi]$ of (periodic) square integrable functions where

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x \quad \text{and} \quad \|f\| = \sqrt{\langle f, f\rangle}$$

are the scalar product and norm, respectively. $L^2[-\pi,\pi]$ is again a Hilbert space.

Theorem 2.7 *The family of functions*

$$\{e^{-ik\omega} \ : \ k \in \mathbb{Z}\}$$

forms an orthonormal basis of $L^2[-\pi,\pi]$.

Proof: Have a look in your favourite Analysis book. □

The above result implies that any function $f \in L^2[-\pi,\pi]$ can be approximated by a Fourier series

$$f(\omega) = \sum_{k=-\infty}^{\infty} \hat{f}[k]e^{-i\omega k}$$

with Fourier coefficients

$$\hat{f}[k] = \langle f(\cdot), e^{-ik\cdot}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega)e^{ik\omega}\,\mathrm{d}\omega.$$

Analogous to Theorem 1.8 we have the Plancherel identity:

$$\sum_{k=-\infty}^{\infty} |f[k]|^2 = \|f\|_{l^2}^2 = \|\hat{f}\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{f}(\omega)|^2\,\mathrm{d}\omega.$$

2.5 Discrete Fourier Transform (DFT)

In practice we obtain only a finite number of sample points, say $k = 0, \dots, N-1$. To get a "complete f" we consider N-periodic signals f with

$$f[k] = \tilde{f}[k \bmod N].$$

The **circular convolution** is defined by

$$\boxed{(f \circledast g)[n] := \sum_{k=0}^{N-1} f[k]g[n-k]}$$

for two periodic signals f and g. Let $\vec{f}, \vec{g} \in \mathbb{R}^N$ then we can rewrite the convolution as a matrix-vector product:

$$(\vec{f} \circledast \vec{g})[n] = G\vec{f}[n], \tag{2.11}$$

where

$$G = \begin{bmatrix} g[0] & g[N-1] & \dots & \dots & g[1] \\ g[1] & g[0] & g[N-1] & & \vdots \\ \vdots & g[1] & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & g[N-1] \\ g[N-1] & \dots & \dots & g[1] & g[0] \end{bmatrix}. \tag{2.12}$$

G is a structured matrix, a so called **circulant** which is a special form of a **Toeplitz matrix**.

Applying G on the vector

$$\vec{v} = \left[e^{2\pi i \frac{kn}{N}}\right]_{k=0}^{N-1}$$

for some $n \in \{0, \dots, N-1\}$ yields

$$G\vec{v} = \left[\sum_{k=0}^{N-1} e^{2\pi i \frac{kn}{N}} g[m-k]\right]_{m=0}^{N-1} = \left[\sum_{l=0}^{N-1} e^{2\pi i \frac{n(m-l)}{N}} g[l]\right]_{m=0}^{N-1} = \hat{g}[n] \left[e^{2\pi i \frac{mn}{N}}\right]_{m=0}^{N-1},$$

where

$$\hat{g}[n] = \sum_{l=0}^{N-1} e^{2\pi i \frac{ln}{N}} g[l]$$

Hence, any vector $[e^{2\pi i \frac{kn}{N}}]_{k=0}^{N-1}$ is an eigenvector of G associated with the eigenvalue $\hat{g}[n]$.

Theorem 2.8 *The family* $e_k[n] = e^{2\pi i \frac{kn}{N}}$, $k = 0, \dots, N-1$ *forms an orthogonal basis in the space of signals with period* N.

Proof: Using the hermitian scalar product for $\mathbb{C}^N$ we get

$$\langle e_k, e_l \rangle = \sum_{n=0}^{N-1} e^{2\pi i \frac{kn}{N}} e^{-2\pi i \frac{ln}{N}} = \sum_{n=0}^{N-1} e^{2\pi i \frac{(k-l)n}{N}}.$$

If $k = l$ then the last term equals to $\sum_{n=0}^{N-1} 1 = N$. In the case of $k \neq l$ we get $\langle e_k, e_l \rangle = 0$, because using geometric series gives

$$\sum_{n=0}^{N-1} e^{2\pi i \frac{(k-l)n}{N}} = \frac{e^{2\pi i \frac{(k-l)N}{N}} - 1}{e^{2\pi i \frac{k-l}{N}} - 1} = 0.$$

Therefore

$$\langle e_k, e_l \rangle = \begin{cases} N & \text{if } k = l \\ 0 & \text{otherwise} \end{cases} = N\delta_{k,l}.$$

□

This implies that any signal f of period N can be decomposed with respect to this basis:

$$f = \sum_{k=0}^{N-1} \langle f, e_k \rangle \frac{e_k}{\|e_k\|^2}.$$

Definition 2.9 *The* **discrete Fourier transform (DFT)** *of a vector $f \in \mathbb{C}^n$ is defined by*

$$\boxed{\hat{f}[k] = \langle f, e_k \rangle = \sum_{n=0}^{N-1} f[n] e^{-2\pi i \frac{kn}{N}}.}$$

As in the convolution case we can rewrite this formula in matrix language:

$$\hat{f} = F^N \cdot f, \tag{2.13}$$

where

$$F^N = \left[f^N_{pk}\right]_{p,k=0}^{N-1} = \left[e^{2\pi i \frac{pk}{N}}\right]_{p,k=0}^{N-1}.$$

Since $\|e_k\|^2 = N$ and $(F^N)^{-1} = \frac{1}{N}(F^N)^H$ we obtain the inverse DFT by

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}[k] e^{2\pi i \frac{kn}{N}}$$

and the Plancherel formula

$$\|f\|^2 = \sum_{n=0}^{N-1} |f[n]|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |\hat{f}[k]|^2 = \frac{1}{N} \|\hat{f}\|^2.$$

Remark 2.10

1. The discrete Fourier transform may be considered as a solution of the following interpolation problem:
   For a given function $f \in C^0[0, 2\pi]$ find n coefficients $\alpha_k \in \mathbb{C}$ such that for all $n = 0, \ldots, N-1$:
   $$f\left(\frac{2\pi n}{N}\right) = \sum_{k=0}^{N-1} \alpha_k e^{2\pi i \frac{kn}{N}}.$$

The above conclusions tell us that

$$\alpha_k = \frac{1}{N}\hat{f}[k] = \frac{1}{N}\sum_{k=0}^{N-1} f\left(\frac{2\pi n}{N}\right) e^{-2\pi i \frac{kn}{N}}.$$

2. If a_k denotes the Fourier coefficient

$$a_k = \int_0^{2\pi} f(t)e^{-2\pi i k t}\,\mathrm{d}t,$$

then α_k is an approximation of a_k using the trapezoidal rule for computing the integral.

2.6 Fast Fourier Transform (FFT)

The computation of the vector $[\hat{f}[k]]_{k=0}^{N-1}$ by a naive matrix-vector product implementation requires $2n^2$ flops since the matrix

$$F^N = \left[e^{2\pi i \frac{pk}{N}}\right]_{p,k=0}^{N-1}$$

is fully populated. This complexity can be drastically reduced by the following observation:

Let N be a power of 2. For even coefficients $\hat{f}[2k],\ k = 0, \ldots, N/2-1$, we observe that

$$e^{-2\pi i \frac{2kn}{N}} = e^{-2\pi i \frac{2k(n+N/2)}{N}}.$$

Therefore the computation of these coefficients is reduced to

$$\hat{f}[2k] = \sum_{n=0}^{N/2-1} (f[n] + f[n+N/2])e^{-2\pi i \frac{kn}{N/2}} = F^{N/2} f_e, \tag{2.14}$$

where

$$f_e = [f[n] + f[n+N/2]]_{n=0}^{N/2-1}.$$

For odd indices $[2k+1]$ we obtain by analogous observations

$$\hat{f}[2k+1] = \sum_{n=0}^{N/2-1} e^{\frac{-2\pi i n}{N}}(f[n] - f[n+N/2])e^{-2\pi i \frac{kn}{N/2}} = F^{N/2} f_o, \tag{2.15}$$

where

$$f_o = \left[e^{\frac{-2\pi i n}{N}}(f[n] - f[n+N/2])\right]_{n=0}^{N/2-1}.$$

To compute $F^{N/2}f_e$ (resp. $F^{N/2}f_o$) we proceed by recursion, applying the same ideas as above. We end up with the following algorithm of complexity $O(n\log n)$.

Algorithm 2.11 Fast Fourier Transform

Purpose Given a vector f of length $N = 2^t$, this routine computes $F^N f$ where $F^N = [e^{2\pi i \frac{pk}{N}}]_{p,k=0}^{N-1}$.

function $y = FFT(f, N)$
  if $n \leq 1$ **then**
    $y = f$
  else
    $m = N/2;\ \omega = e^{-2\pi i/N}$
    $y_e = FFT(f[0\!:\!2\!:\!N], m);\ y_o = FFT(f[1\!:\!2\!:\!N], m)$
    $d = [1, \omega, \ldots, \omega^{m-1}]^T$
    $z = d. * y_o$
    $y = \begin{bmatrix} y_e + z \\ y_e - z \end{bmatrix}$
  end

Remark 2.12 Algorithm 2.11 is one of the standard routines which is included in almost any numerical package. A brief overview of the related functions implemented in Matlab:

```
fft    - Discrete Fourier transform
fft2   - Two-dimensional discrete Fourier Transform
ifft   - Inverse discrete Fourier transform
ifft2  - Two-dimensional inverse discrete Fourier transform
conv   - Convolution and polynomial multiplication
conv2  - Two dimensional convolution
deconv - Deconvolution and polynomial division
filter - One-dimensional digital filter
```

Further recommended toolboxes are Mathworks signal processing toolbox and WaveLab which is available via

`http://www-stat.stanford.edu/~wavelab/`

3 Time meets Frequency

We consider so called **time / frequency atoms** with small energy spread in the time/frequency domains. They form a family of functions

$$\{\Phi_\gamma,\ \gamma \in \Gamma\},$$

where γ is a multi index parameter. We suppose $\Phi_\gamma \in L^2(\mathbb{R})$ and normalize such that $\|\Phi_\gamma\|_{L^2} = 1$.

We define a **time/frequency operator** $T : \Gamma \to \mathbb{C}/\mathbb{R}$ for functions $f \in L^2(\mathbb{R})$ by

$$\boxed{Tf(\gamma) := \langle f, \Phi_r \rangle = \int_{-\infty}^{\infty} f(t)\overline{\Phi_\gamma(t)}\ \mathrm{d}t,} \tag{3.1}$$

and reformulate with the help of Parsevals identity (Theorem 1.8)

$$Tf(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\hat{\Phi}_\gamma^*(\xi)\ \mathrm{d}\xi. \tag{3.2}$$

If Φ_γ is nearly zero besides a small interval then (3.1) tell us that $Tf(\gamma)$ depends only on the values of f in this interval. Similarly, if Φ_γ has an approximately small frequency support then from (3.2) we know that $Tf(\gamma)$ reveals the properties of $\hat{f}$ on a small interval.

Example 3.1

1. "Windowed Fourier Transform Atom"

$$\Phi_\gamma = g_{x,\xi}(t) \ := \ g(t-x)e^{i\xi t},$$

   where $g(t) = g(-t) \in \mathbb{R}$ and $\xi, x \in \mathbb{R}$.

2. "Wavelet Atom"

$$\Phi_\gamma(t) = \Psi_{a,x}(t) \ := \ \frac{1}{\sqrt{a}}\Psi\left(\frac{t-x}{a}\right)$$

   Ψ is a so called **wavelet** with zero mean, i.e. $\int_{-\infty}^{\infty}\Psi(x)\ \mathrm{d}x = 0$. $a > 0$ is the **scaling parameter**.

Both are examples for localization in the time variable.

3.1 Heisenberg Boxes

We interpret the function $|\Phi_\gamma(t)|^2$ as a probality density function ($\|\Phi_\gamma\| = 1$). The **center** (resp. expectation) x_γ of $|\Phi_\gamma(t)|^2$ is defined by

$$x_\gamma = \int_{-\infty}^{\infty} t|\Phi_\gamma(t)|^2\ \mathrm{d}t.$$

The **variance** $\sigma_t(\gamma)$ expresses the scattering from the center:

$$\sigma_t^2(\gamma) = \int_{-\infty}^{\infty}(t-x_\gamma)^2|\Phi_\gamma(t)|^2\ \mathrm{d}t.$$

Since $\|\hat{\Phi}_r\|_{L^2}^2 = 2\pi$ we get analog results for the Fourier transform. The frequency center is

$$\omega_\gamma = \frac{1}{2\pi}\int_{-\infty}^{\infty}\xi|\hat{\Phi}(\xi)|^2\ \mathrm{d}\xi,$$

and the variance with respect to frequency

$$\sigma_\xi^2(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\xi-\omega_\gamma)^2|\hat{\Phi}(\xi)|^2\ \mathrm{d}\xi.$$

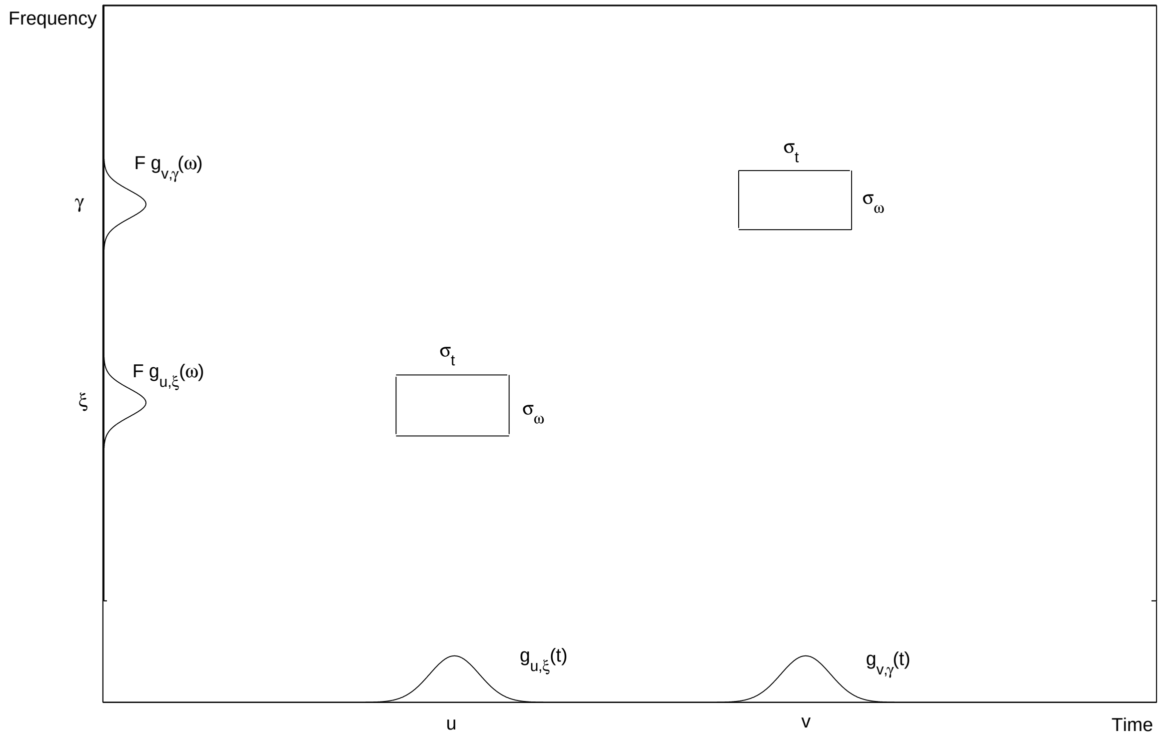


Figure 3.1: Heisenberg Box of two windowed Fourier atoms $g_{u,\xi}$ and $g_{v,\gamma}$.

Now, the time/frequency localization may be expressed by an rectangle in the t-ξ-time-freqency plane:

$$\left[x_\gamma - \frac{\sigma_t(\gamma)}{2}, x_\gamma + \frac{\sigma_t(\gamma)}{2}\right] \times \left[\omega_\gamma - \frac{\sigma_\xi(\gamma)}{2}, \omega_\gamma + \frac{\sigma_\xi(\gamma)}{2}\right]$$

which is the so called **"Heisenberg Box"**. Our desired property is to have Φ_γ well localized in time and frequency. So, optimally σ_t and σ_ξ should be very small values. But we already know from Heisenbergs uncertainty (see Proposition 1.12) that

$$\sigma_t(\gamma) \cdot \sigma_\xi(\gamma) \geq \frac{1}{2}.$$

3.2 Windowed Fourier Transform

We consider as a special time atom the "time window" $g : \mathbb{R} \to \mathbb{R}$ as an even function which is modulated by a frequency ξ:

$$g_{x,\xi}(t) = g(t-x)e^{it\xi}, \quad \|g\| = 1.$$

Consequently we have $\|g_{x,\xi}\| = 1$ and the center of $g_{x,\xi}$ is

$$\int_{-\infty}^{\infty} tg^2(t-x) \, \mathrm{d}t = x$$

with a variance of

$$\sigma_t^2 = \int_{-\infty}^{\infty} (t-x)^2 |g(t-x)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} \tau^2 g^2(\tau) \, \mathrm{d}\tau =: \sigma^2$$

which is independent of x and ξ. For $\hat{g}_{x,\xi}(\omega) = e^{-ix(\omega-\xi)}\hat{g}(\omega-\xi)$ we get the variance

$$\sigma_\xi^2 = \int_{-\infty}^{\infty} (\omega-\xi)^2 |\hat{g}_{x,\xi}(\omega-\xi)|^2 \, \mathrm{d}\omega = \int_{-\infty}^{\infty} \tau^2 |\hat{g}(\tau)|^2 \, \mathrm{d}\tau =: \hat{\sigma}^2$$

independent of x and ξ. The center of $\hat{g}_{x,\xi}$ is ξ.

Definition 3.2 *The* **Windowed-Fourier-Transform (WFT)** *of $f \in L^2(\mathbb{R})$ is defined by*

$$\boxed{Sf(x,\xi) = \langle f, g_{\omega,\xi} \rangle = \int_{-\infty}^{\infty} f(t) e^{-i\xi t} g(t-x) \, \mathrm{d}t,} \tag{3.3}$$

while the **energy density** *is*

$$\boxed{PSf(x,\xi) = |Sf(x,\xi)|^2.}$$

Example 3.3

1. Sinusoidal Wave

$$\begin{aligned} f(t) = e^{i\omega_0 t} \quad &\Rightarrow \quad \hat{f}(\xi) = 2\pi\delta(\xi-\omega_0) \\ \text{WFT} \quad &\Rightarrow \quad Sf(x,\xi) = e^{-ix(\xi-\omega_0)}\hat{g}(\xi-\omega_0) \end{aligned}$$

2. Dirac impuls

$$f(t) = \delta(t-x_0) \;\Rightarrow\; Sf(x,\xi) = e^{ix_0\xi_0} g(x_0-x)$$

Theorem 3.4 *Reconstruction formula*
Let $f \in L^2(\mathbb{R})$. Then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Sf(x,\xi) g(t-x) e^{it\xi} \, \mathrm{d}x \, \mathrm{d}\xi, \tag{3.4}$$

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Sf(x,\xi)|^2 \, \mathrm{d}x \, \mathrm{d}\xi. \tag{3.5}$$

Proof:

$$Sf(x,\xi) = \int_{-\infty}^{\infty} f(t) g(t-x) e^{i\xi(x-t)} e^{-i\xi x} \, \mathrm{d}t = e^{-i\xi x} \int_{-\infty}^{\infty} f(t) g_\xi(x-t) \, \mathrm{d}t,$$

where $g_\xi(t) = g(t)e^{i\xi t}$. Hence

$$Sf(x,\xi) = e^{-i\xi x}(f * g_\xi)(x) =: f_\xi(x). \tag{3.6}$$

Using convolution Theorem 1.5 yields

$$\hat{f}_\xi(\omega) = \hat{f}(\omega+\xi)\hat{g}_\xi(\omega+\xi) = \hat{f}(\omega+\xi)\hat{g}(\omega). \tag{3.7}$$

Recall that the Fourier transform of $g(\cdot - t)$ is $\hat{g}(\omega)e^{-i\omega t}$. Applying Parsevals formula (Theorem 1.8) on the equality (3.4) yields

$$\begin{aligned}
I &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} Sf(x,\xi)g(t-x)\,\mathrm{d}x\right)e^{it\xi}\,\mathrm{d}\xi \\
(3.6) \Rightarrow &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}_\xi(\omega)\overline{\hat{g}(\omega)}e^{it(\xi+\omega)}\,\mathrm{d}\omega\,\mathrm{d}\xi \\
(3.7) \Rightarrow &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(\omega+\xi)|\hat{g}(\omega)|^2 e^{it(\xi+\omega)}\,\mathrm{d}\omega\,\mathrm{d}\xi \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{g}(\omega)|^2\,\mathrm{d}\omega\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega+\xi)e^{it(\xi+\omega)}\,\mathrm{d}\xi\right) \\
&= f(t)\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{g}(\omega)|^2\,\mathrm{d}\omega = \frac{1}{2\pi}f(t)\|\hat{g}\|^2 = f(t)\|g\|^2 = f(t).
\end{aligned}$$

To prove (3.5) we use the same technique as above:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Sf(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{f}_\xi(\omega)|^2\,\mathrm{d}\omega\,\mathrm{d}\xi \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{f}(\xi+\omega)|^2|\hat{g}(\omega)|^2\,\mathrm{d}\xi\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\|\hat{f}\|^2\frac{1}{2\pi}\|\hat{g}\|^2 = \|f\|^2.
\end{aligned}$$

□

The reconstruction formula may be rewritten as

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle f, g_{x,\xi}\rangle g_{x,\xi}\,\mathrm{d}\xi\,\mathrm{d}x.}$$

Theorem 3.5 *Let $\Phi(\cdot,\cdot) \in L^2(\mathbb{R}^2)$. Then there exists a function $f \in L^2(\mathbb{R})$ with $\Phi(x,\xi) = Sf(x,\xi)$ if and only if*

$$\Phi(x_0,\xi_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(x,\xi)K(x,x_0,\xi,\xi_0)\,\mathrm{d}x\,\mathrm{d}\xi, \tag{3.8}$$

where

$$K(x,x_0,\xi,\xi_0) = \langle g_{x,\xi}, g_{x_0,\xi_0}\rangle = \int_{-\infty}^{\infty} g(t-x)e^{i\xi t}\overline{g(t-x_0)}e^{-i\xi_0 t}\,\mathrm{d}t$$

holds for all $(t_0,\xi_0) \in \mathbb{R}^2$).

Proof: Suppose there exists an $f \in L^2(\mathbb{R})$ with $\Phi(x,\xi) = Sf(x,\xi)$, then

$$\begin{aligned}\Phi(x_0,\xi_0) &= Sf(x_0,\xi_0) = \int_{-\infty}^{\infty} f(t)\overline{g(t-x_0)}e^{-i\xi_0 t}\,\mathrm{d}t \\ \text{(Theorem 3.4)} \Rightarrow &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{g(t-x_0)}e^{-i\xi_0 t}\Phi(x,\xi)g(t-x)e^{i\xi t}\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t.\end{aligned}$$

Using Fubinis theorem proves (3.8).

Conversely, define

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-t)e^{i\xi t}\Phi(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi,$$

and $Sf(x_0,\xi_0) = \Phi(x_0,\xi_0)$ results from the same considerations as above. □

3.3 Wavelet Transform

Definition 3.6 *$\Psi \in L^2(\mathbb{R})$ is called a* **wavelet** *if $\|\Psi\| = 1$ and $\int_{-\infty}^{\infty}\Psi(x)\,\mathrm{d}x = 0$. We regard Ψ to be* **centered** *if the expectation $\int_{-\infty}^{\infty} t|\Psi(t)|^2\,\mathrm{d}t$ is finite. For $x \in \mathbb{R}$ and $a > 0$ a shifted and scaled wavelet version is defined by*

$$\boxed{\Psi_{a,x}(t) := \frac{1}{\sqrt{a}}\Psi\left(\frac{t-x}{a}\right).} \tag{3.9}$$

Definition 3.7 *The* **continuous wavelet transform (WT)** *of $f \in L^2(\mathbb{R})$ at the time x and scale $a > 0$ is defined by*

$$\boxed{Wf(x,a) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{a}}\Psi^*\left(\frac{t-x}{a}\right)\,\mathrm{d}t = \langle f, \Psi_{a,x}\rangle.} \tag{3.10}$$

We can reformulate (WT) as a convolution

$$Wf(x,a) = \left(f * \tilde{\Psi}_a\right)(x), \tag{3.11}$$

where

$$\tilde{\Psi}_a(t) = \frac{1}{\sqrt{a}}\Psi^*\left(-\frac{t}{a}\right).$$

Since

$$\widehat{Wf(\cdot,a)}(\xi) = \hat{f}(\xi)\hat{\tilde{\Psi}}_a(\xi) = \sqrt{a}\hat{f}(\xi)\hat{\Psi}^*(a\xi) \tag{3.12}$$

and $\hat{\Psi}(0) = \int_{-\infty}^{\infty}\Psi(x)\,\mathrm{d}x = 0$, the (WT) acts as a band filter with scaling a.

3.3.1 Real Wavelets

A real wavelet transform is complete and satsifies an energy conservation, as long as a weak condition is satisfied, as the following theorem shows.

Theorem 3.8 'Calderón, Grossmann, Morlet'
Let $\Psi \in L^2(\mathbb{R})$ *be real-valued with* $\int_{-\infty}^{\infty} \Psi(x)\ \mathrm{d}x = 0$ *and*

$$C_\Psi = \int_0^\infty \frac{|\hat{\Psi}(\omega)|^2}{\omega}\ \mathrm{d}\omega < \infty \tag{3.13}$$

which is the **Wavelet admissibility condition**. *Then we have*

$$f(t) = \frac{1}{C_\Psi}\int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{a}} Wf(x,a)\Psi\left(\frac{x-t}{a}\right)\ \mathrm{d}x\frac{\mathrm{d}a}{a^2}, \tag{3.14}$$

$$\|f\|^2 = \frac{1}{C_\Psi}\int_0^\infty \int_{-\infty}^\infty |Wf(x,\xi)|^2\ \mathrm{d}x\frac{\mathrm{d}a}{a^2}. \tag{3.15}$$

Proof: We use the convolution formulation (3.11), $Wf(x,a) = (f * \tilde{\Psi}_a)(x)$. Let

$$b(t) = \frac{1}{C_\Psi}\int_0^\infty (Wf(\cdot,a) * \Psi_a)(t)\frac{\mathrm{d}a}{a^2},$$

then with (3.12) and Fubinis theorem

$$\begin{aligned} \hat{b}(\xi) &= \frac{1}{C_\Psi}\int_0^\infty \hat{f}(\xi)\sqrt{a}\hat{\Psi}^*(a\xi)\sqrt{a}\hat{\Psi}(a\xi)\frac{\mathrm{d}a}{a^2} \\ (\tau = a\xi) \Rightarrow &= \frac{\hat{f}(\xi)}{C_\Psi}\int_0^\infty |\hat{\Psi}(\tau)|^2\frac{\mathrm{d}\tau}{\tau} = \hat{f}(\xi). \end{aligned}$$

Hence $b(t) = f(t), \forall t \in \mathbb{R}$ and (3.14) is proven. Formula (3.15) results from similar considerations as they were outlined in the proof of (3.5) in Theorem 3.4. □

Example 3.9 The so called **Mexican hat wavelet** (see also Figure 3.2) is defined as

$$\Psi(t) = \frac{2}{\pi^{\frac{1}{4}}\sqrt{3\sigma}}\left(\frac{t^2}{\sigma^2} - 1\right)e^{-\frac{t^2}{2\sigma^2}} \in \mathbb{R},$$

and its Fourier transform is given by

$$\hat{\Psi}(\xi) = -\frac{\sqrt{8}\sigma^{\frac{5}{2}}\pi^{\frac{1}{4}}}{\sqrt{3}}\xi^2 e^{-\frac{\xi^2\sigma^2}{2}} \in \mathbb{R}.$$

For continuous functions $\hat{\Psi}$ the condition $\hat{\Psi}(0) = 0$ or equivalently $\int_{-\infty}^{\infty} \Psi(t)\ \mathrm{d}t = 0$ is necessary for the Wavelet admissibility condition. In other words, the first moment of Ψ is zero. If furthermore $\hat{\Psi} \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ holds, then the admissibility is guaranteed. The property $\hat{\Psi} \in C^1(\mathbb{R})$ is satisfied e.g. if

$$\int_{-\infty}^{\infty} (1 + |x|)|\Psi(x)|\ \mathrm{d}x < \infty.$$

Reproducing Kernels

Applying the Wavelet transformation on the representation formula (3.14) yields

$$\begin{aligned} Wf(x,y) &= \int_{-\infty}^{\infty}\left(\frac{1}{C_\Psi}\int_0^\infty\int_{-\infty}^{\infty}\frac{Wf(u,s)}{\sqrt{s}}\Psi\left(\frac{t-u}{s}\right)\,\mathrm{d}u\frac{\mathrm{d}s}{s^2}\right)\frac{1}{\sqrt{y}}\Psi^*\left(\frac{t-x}{y}\right)\,\mathrm{d}t \\ &= \int_0^\infty\int_{-\infty}^{\infty}\frac{Wf(u,s)}{C_\Psi}\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{sy}}\Psi\left(\frac{t-u}{s}\right)\Psi^*\left(\frac{t-x}{y}\right)\,\mathrm{d}t\right)\,\mathrm{d}u\frac{\mathrm{d}s}{s^2} \\ &= \int_0^\infty\int_{-\infty}^{\infty}\frac{1}{C_\Psi\sqrt{sy}}\left\langle\Psi\left(\frac{\cdot-u}{s}\right),\Psi\left(\frac{\cdot-x}{y}\right)\right\rangle Wf(u,s)\,\mathrm{d}u\frac{\mathrm{d}s}{s^2}. \end{aligned}$$

Setting

$$K(x,u;\ y,s) = \frac{1}{\sqrt{sy}}\left\langle\Psi\left(\frac{\cdot-u}{s}\right),\Psi\left(\frac{\cdot-x}{y}\right)\right\rangle$$

we obtain the **reproducing kernel equation**

$$Wf(x,y) = \frac{1}{C_\Psi}\int_0^\infty\int_{-\infty}^{\infty}K(x,u;\ y,s)Wf(u,s)\,\mathrm{d}u\frac{\mathrm{d}s}{s^2}. \tag{3.16}$$

Note that $K(x,u;\ y,s)$ can be considered as the wavelet analogon of the WFT kernel (see Theorem 3.5). The modulus of the reproducing kernel $K(x,u;\ y,s)$ is a measure for the correlation between two wavelets $\Psi_{x,y}$ and $\Psi_{u,s}$. The diligent reader may verify that any function $\Phi(u,s)$ is the wavelet transform of some $f \in L^2(\mathbb{R})$ if and only if it satisfies equation (3.16).

Scaling Functions

When $Wf(u,s)$ is only known for $s < s_0$, what can we say about the contained information ?

Let us introduce the **scaling function** $\varphi \in L^2(\mathbb{R})$ by the modulus of its Fourier transform

$$|\hat{\varphi}(\omega)|^2 = \int_1^\infty|\hat{\Psi}(s\omega)|^2\frac{\mathrm{d}s}{s} = \int_\omega^\infty|\hat{\Psi}(u)|^2\frac{\mathrm{d}u}{u}.$$

The complex phase of $\hat{\varphi}$ can be abritrarily chosen. It can be shown that

$$\|\varphi\| = 1 \quad\text{and}\quad \lim_{\omega\to 0}|\hat{\varphi}(\omega)|^2 = C_\Psi.$$

We denote $\varphi_s(t) = \frac{1}{\sqrt{s}}\varphi\left(\frac{t}{s}\right)$ and $\tilde{\varphi}_s(t) = \overline{\varphi_s(-t)}$ so that for $f \in L^2(\mathbb{R})$ the low-frequency approximation of f at scale s is

$$Lf(u,s) = \left\langle f,\frac{1}{\sqrt{s}}\,\varphi\left(\frac{\cdot-u}{s}\right)\right\rangle = (f*\tilde{\varphi}_s)(u).$$

With a minor modification of the proof of the reconstruction formula in Theorem 3.4 one can show that

$$f(t) = \frac{1}{C_\Psi}\int_0^{s_0}(W(\cdot,s)*\Psi_s)(t)\frac{\mathrm{d}s}{s^2} + \frac{1}{C_\Psi s_0}(Lf(\cdot,s_0)*\varphi_{s_0})(t). \tag{3.17}$$

Example 3.10 If Ψ is again the Mexican hat wavelet from Example 3.9, then

$$\hat{\varphi}(\omega) = \frac{2\sigma^{\frac{3}{2}}\pi^{\frac{1}{4}}}{\sqrt{3}}\sqrt{\omega^2+\frac{1}{\sigma^2}}e^{-\frac{\sigma^2\omega^2}{2}}.$$

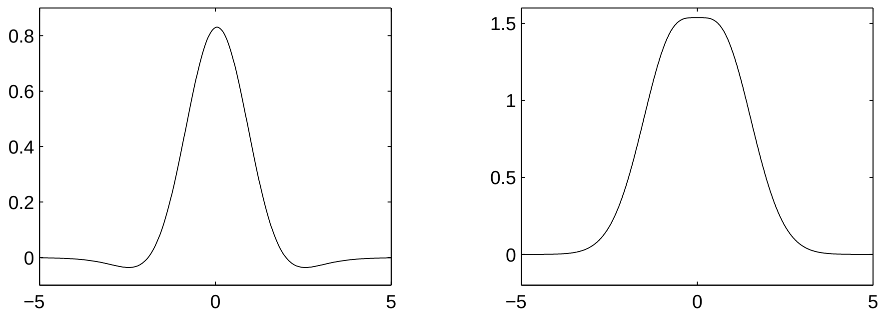


Figure 3.2: Scaling function associated with a Mexican hat wavelet and its Fourier transform for $\sigma = 1$.

3.3.2 Analytic Wavelets

Definition 3.11 *We call a signal* $f_a \in L^2(\mathbb{R})$ **analytic** *if its Fourier transform satisfies* $\hat{f}_a(\omega) = 0$ *for all* $\omega < 0$.

An analytic function f_a is in general complex but completely characterized by its real part $f(t) = \text{Re } f_a(t)$. The Fourier transform of f is given by

$$\hat{f}(\omega) = \frac{\hat{f}_a(\omega) + \overline{\hat{f}_a(-\omega)}}{2} \tag{3.18}$$

and vice versa

$$\hat{f}_a(\omega) = \begin{cases} 2\hat{f}(\omega) & \text{for } \omega \geq 0, \\ 0 & \text{for } \omega < 0. \end{cases} \tag{3.19}$$

Example 3.12 The Fourier transform of

$$f(t) = a\cos(\omega_0 t + \varphi) = \frac{a}{2}\left(e^{i(\omega_0 t+\varphi)} + e^{-i(\omega_0 t+\varphi)}\right)$$

is

$$\hat{f} = \pi\left(e^{i\varphi}\delta(\omega - \omega_0) + e^{-i\varphi}\delta(\omega + \omega_0)\right).$$

Its analytic part satsifies $\hat{f}_a(\omega) = 2\pi a e^{i\varphi}\delta(\omega - \omega_0)$ and hence

$$f_a(t) = ae^{i(\omega_0 t+\varphi)}.$$

Definition 3.13 *Let* Ψ *be an analytic Wavelet, then the* **analytic Wavelet transform** *is defined by*

$$\boxed{Wf(u,s) = \langle f, \Psi_{s,u}\rangle = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*\left(\frac{t-u}{s}\right)\,\mathrm{d}t.}$$

Suppose Ψ to be centered at 0, i.e. $\int_{-\infty}^{\infty} t|\Psi(t)|^2 \, \mathrm{d}t = 0$, and define

$$\sigma_t^2 := \int_{-\infty}^{\infty} t^2 |\Psi(t)|^2 \, \mathrm{d}t.$$

Then $\Psi_{s,u}$ is centered at $u \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} (t-u)^2 \left| \frac{1}{\sqrt{s}} \Psi\left(\frac{t-u}{s}\right) \right|^2 \, \mathrm{d}t = s^2 \sigma_t^2 =: \tilde{\sigma}_t^2.$$

Let $\eta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega |\hat{\Psi}(\omega)|^2 \, \mathrm{d}\omega$ be the center frequency of $\hat{\Psi}$ and

$$\sigma_\omega^2 = \int_{-\infty}^{\infty} (\omega - \eta)^2 |\hat{\Psi}(\omega)|^2 \, \mathrm{d}\omega.$$

The Fourier transform of $\Psi_{u,s}$ is a dilation of $\hat{\Psi}$ by $1/s$:

$$\hat{\Psi}_{u,s}(\omega) = \sqrt{s} \hat{\Psi}(s\omega) e^{-i\omega u}. \tag{3.20}$$

Its center frequency is therefore η/s. The energy spread of $\hat{\Psi}_{u,s}$ around η/s is

$$\int_{-\infty}^{\infty} \left(\omega - \frac{\eta}{s}\right)^2 s |\hat{\Psi}(s\omega)|^2 \, \mathrm{d}\omega = \frac{\sigma_\omega^2}{s^2} =: \tilde{\sigma}_\omega^2$$

Hence $\tilde{\sigma}_t^2 \tilde{\sigma}_\omega^2 = \sigma_t^2 \sigma_\omega^2$ is constant. Thus, the energy spread of a wavelet-time frequency atom $\Psi_{u,s}$ corresponds to a $s\sigma_t \times \frac{\sigma_\omega}{s}$ Heisenberg box centered at $(u, \frac{\eta}{s})$ with constant area.

$$P_W f(u, \xi) := |Wf(u, s)|^2 \tag{3.21}$$

is the so called **scalogram** of f and represents the energy density.

The following theorem derives a reconstruction formula for the wavelet transform and proves that energy is preserved for real signals.

Theorem 3.14 *Let Ψ be an analytic wavelet, then for any $f \in L^2(\mathbb{R})$*

$$\boxed{Wf(u, s) = \frac{1}{2} Wf_a(u, s).} \tag{3.22}$$

If Ψ is admissible (i.e. $C_\Psi < \infty$, see (3.13) in Theorem 3.8) and f real valued, then

$$f(t) = \frac{2}{C_\Psi} Re \left[\int_0^{\infty} \int_{-\infty}^{\infty} Wf(u, s) \frac{1}{\sqrt{s}} \Psi\left(\frac{u-t}{s}\right) \, \mathrm{d}u \frac{\mathrm{d}s}{s^2} \right], \tag{3.23}$$

and

$$\|f\|^2 = \frac{2}{C_\Psi} \int_0^{\infty} \int_{-\infty}^{\infty} |Wf(u, s)|^2 \, \mathrm{d}u \frac{\mathrm{d}s}{s^2}. \tag{3.24}$$

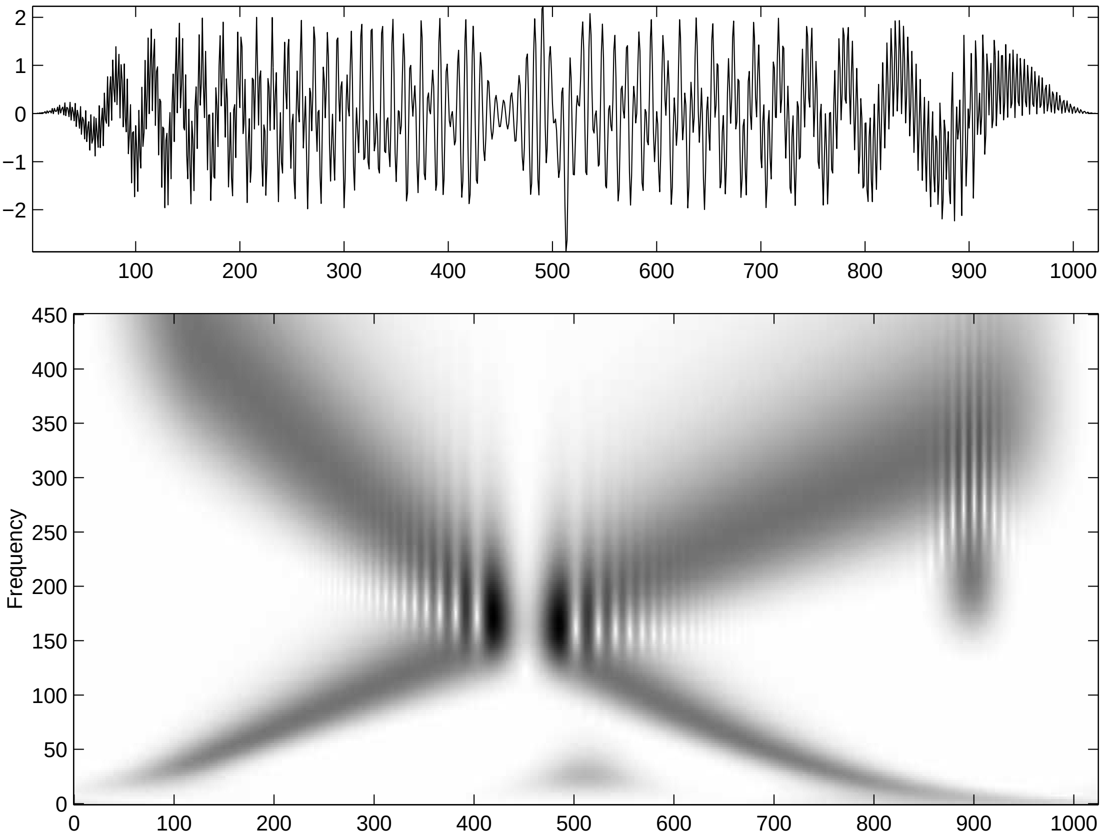


Figure 3.3: The top signal includes a linear chirp whose frequency increases, a quadratic chirp whose frequency decreases, and two modulated Gaussian functions located at $t = 512$ and $t = 896$. The bottom image is the scalogram.

Proof: We first prove (3.22). The Fourier transform of $f_s(u) = Wf(u,s) = (f * \bar{\Psi}_s)(u)$ with $s > 0$ is

$$\hat{f}_s(\omega) = \hat{f}(\omega)\hat{\Psi}_s^*(\omega) = \hat{f}(\omega)\sqrt{s}\hat{\Psi}^*(s\omega).$$

Since $\hat{\Psi}(\omega) = 0$ for negative frequencies, and $\hat{f}_a = 2\hat{f}(\omega)$ we get

$$\hat{f}_s(\omega) = \frac{1}{2}\hat{f}_a(\omega)\sqrt{s}\hat{\Psi}^*(\omega s) = \frac{1}{2}\widehat{Wf_a(\cdot,s)}(\omega),$$

which proves (3.22).

With the same derivations as done in the proof of (3.14) in Theorem 3.8 one can show that

$$f_a(t) = \frac{1}{C_\Psi}\int_0^\infty\int_{-\infty}^\infty Wf_a(u,s)\frac{1}{\sqrt{s}}\Psi\left(\frac{u-t}{s}\right)\,\mathrm{d}u\frac{\mathrm{d}s}{s^2}.$$

Since $f = \mathrm{Re} f_a$, inserting (3.22) proves (3.23).

Analogous we claim that

$$\|f_a\|^2 = \frac{1}{C_\Psi}\int_0^\infty\int_{-\infty}^\infty |Wf_a(u,s)|^2\,\mathrm{d}u\frac{\mathrm{d}s}{s^2} = \frac{4}{C_\Psi}\int_0^\infty\int_{-\infty}^\infty |Wf(u,s)|^2\,\mathrm{d}u\frac{\mathrm{d}s}{s^2}.$$

Parsevals identity and formula (3.18) give $\|f_a\|^2 = \frac{1}{2\pi}\|\hat{f}_a\|^2 = \frac{1}{\pi}\|\hat{f}\|^2 = 2\|f\|^2$ which completes the proof. □

Wavelet Modulated Windows

An analytic wavelet can be constructed with a frequency modulation of a real and symmetric window g. The Fourier transform of

$$\Psi(t) = e^{i\eta t} g(t)$$

is $\hat{\Psi}(\omega) = \hat{g}(\omega - \eta)$. If $\hat{g}(\omega) = 0$ for $|\omega| > \eta$ then $\hat{\Psi}(\omega) = 0$ for all $\omega < 0$. Hence Ψ is analytic. The center frequency of $\hat{\Psi}$ is η and

$$|\hat{\Psi}(\eta)| = \sup_{\omega\in\mathbb{R}} |\hat{\Psi}(\omega)| = \hat{g}(0).$$

Example 3.15 'Gabor Wavelet'
Consider the Gaussian window

$$g(t) = \frac{1}{(\sigma^2\pi)^{1/4}} e^{-\frac{t^2}{2\sigma^2}} \quad \Rightarrow \quad \hat{g}(\omega) = (4\pi\sigma^2)^{1/4} e^{-\frac{\sigma^2\omega^2}{2}}.$$

If $\sigma^2\eta^2 \gg 1$ then $\hat{g}(\omega) \approx 0$ for $|\omega| > \eta$. Sufficiently large η supply approximately analytic wavelets $\Psi(t) = g(t)e^{i\eta t}$, the so called Gabor Wavelets.

4 Time-Frequency Energy

The wavelet and windowed Fourier transforms are computed by correlating the signal with families of time-frequency atoms. The time and frequency resolution of these transforms is thus limited by the time-frequency resolution of the corresponding atoms. Ideally, one would like to define a density of energy in a time-frequency plane, which does not spread the signal energy in time or in frequency.

The Wigner-Ville distribution is a time-frequency energy density computed by correlating f with a time and frequency translation by itself. This avoids any loss of time-frequency resolution.

4.1 Wigner-Ville Distribution

Definition 4.1 *Let $f \in L^2(\mathbb{R})$. The* **Wigner-Ville distribution** *of f is defined by*

$$P_V f(u,\xi) = \int_{-\infty}^{\infty} f\left(u + \frac{\tau}{2}\right) f^*\left(u - \frac{\tau}{2}\right) e^{-i\xi\tau}\, \mathrm{d}\tau \tag{4.1}$$

$$= \left(f\left(u + \frac{\cdot}{2}\right) f^*\left(u - \frac{\cdot}{2}\right)\right)^{\wedge}(\xi). \tag{4.2}$$

The Wigner-Ville distribution remains real and using Parsevals formula (4.1) it can be rewritten as

$$P_V f(u,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\left(\xi + \frac{\omega}{2}\right) \hat{f}^*\left(\xi - \frac{\omega}{2}\right) e^{i\omega u}\, \mathrm{d}\omega. \tag{4.3}$$

Proposition 4.2 *Let $f \in L^2(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} P_V f(u,\xi)\ \mathrm{d}u = |\hat{f}(\xi)|^2, \tag{4.4}$$

and

$$\int_{-\infty}^{\infty} P_V f(u,\xi)\ \mathrm{d}\xi = 2\pi |f(u)|^2. \tag{4.5}$$

Proof: Let $\hat{g}_u(\xi) = P_V f(u,\xi)$. Applying the inverse Fourier transform on (4.2) yields

$$g_u(t) = f\left(u + \frac{t}{2}\right) \overline{f\left(u - \frac{t}{2}\right)},$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} P_V f(u,\xi)\ \mathrm{d}\xi &= \int_{-\infty}^{\infty} \hat{g}_u(\xi)\ \mathrm{d}\xi = 2\pi g_u(0) \\ &= 2\pi f(u)\overline{f(u)} = 2\pi |f(u)|^2,\end{aligned}$$

which proves (4.5). For the first equality we consider $g_\xi(u) = P_V f(u,\xi)$ where (4.3) supplies the suitable Fourier transform

$$\hat{g}_\xi(\omega) = \hat{f}\left(\xi + \frac{\omega}{2}\right) \hat{f}^*\left(\xi - \frac{\omega}{2}\right).$$

Hence, (4.4) is proven by

$$\int_{-\infty}^{\infty} g_\xi(u)\ \mathrm{d}u = \hat{g}_\xi(0) = |\hat{f}(\xi)|^2.$$

□

The Wigner-Ville distribution is a quadratic form, it may be considered as an energy density in the time-frequency plane. However, it is lacking one fundamental property of an energy density, namely **positivity**, which demonstrates the following example.

Example 4.3 The W-V distribution of $f(t) = \xi_{[-T,T]}$ is

$$\begin{aligned}P_V f(u,\xi) &= \int_{-\infty}^{\infty} f\left(u + \frac{\tau}{2}\right) f\left(u - \frac{\tau}{2}\right) e^{-i\xi\tau}\ \mathrm{d}\tau \\ &= 2\frac{\sin(2(T-|u|)\xi)}{\xi}.\end{aligned}$$

This is an oscillating function that takes negative values.

The following proposition shows that the W-V distribution does not alter time and frequency supports of f and $\hat{f}$, respectively.

Proposition 4.4 *If f has compact support, i.e. supp $f \subseteq \left[u_0 - \frac{T}{2}, u_0 + \frac{T}{2}\right]$, then*

$$\text{supp } P_V f(u,\xi) \subseteq \left[u_0 - \frac{T}{2}, u_0 + \frac{T}{2}\right] \times \mathbb{R}, \tag{4.6}$$

and if supp $\hat{f} \subseteq \left[\omega_0 - \frac{\Omega}{2}, \omega_0 + \frac{\Omega}{2}\right]$, then

$$\text{supp } P_V f(u,\xi) \subseteq \mathbb{R} \times \left[\omega_0 - \frac{\Omega}{2}, \omega_0 + \frac{\Omega}{2}\right]. \tag{4.7}$$

Proof: Let $\tilde{f}(t) = f(-t)$, then

$$P_V f(u,\xi) = \int_{-\infty}^{\infty} f\left(\frac{\tau + 2u}{2}\right) \tilde{f}^*\left(\frac{\tau - 2u}{2}\right) e^{-i\xi\tau}\, \mathrm{d}\tau, \tag{4.8}$$

and for the supports of f and $\tilde{f}$ we have

$$\begin{aligned} \text{supp } f\left(\frac{\cdot + 2u}{2}\right) &\subseteq [2(u_0 - u) - T, 2(u_0 - u) + T], \\ \text{supp } \tilde{f}\left(\frac{\cdot - 2u}{2}\right) &\subseteq [-2(u_0 + u) - T, -2(u_0 + u) + T]. \end{aligned}$$

The Wigner-Ville integral (4.8) is zero if these two supports do not overlap, which is the case only if $|u_0 - u| < \frac{T}{2}$. (4.7) can be proven in a similar way using the following alternative representation of the W-V distribution

$$P_V f(u,\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}\left(\frac{\omega + 2\xi}{2}\right) \widehat{\tilde{f}}^*\left(\frac{\omega - 2\xi}{2}\right) e^{i\omega u}\, \mathrm{d}\omega.$$

□

Example 4.5 Proposition 4.4 shows that the Wigner-Ville distribution does not spread the time or frequency support of Diracs or sinusoids, unlike windowed Fourier or wavelet transforms. In fact we get

$$\begin{aligned} f(t) = \delta(t - u_0) &\Rightarrow P_V f(u,\xi) = \delta(u - u_0), \\ f(t) = e^{i\omega_0 t} &\Rightarrow P_V f(u,\xi) = \frac{1}{2\pi}\delta(\xi - \omega_0). \end{aligned}$$

Let

$$f(t) = \frac{1}{(\sigma^2\pi)^{1/4}} e^{-\frac{t^2}{2\sigma^2}}$$

be a Gaussian function, then its Fourier transform $\hat{f}$ is also a Gaussian and

$$P_V f(u,\xi) = \frac{1}{\pi} e^{-\frac{u^2}{\sigma} - \sigma^2\xi^2} = \frac{1}{\pi} e^{-\frac{u^2}{\sigma}} e^{-\sigma^2\xi^2} = |f(u)|^2 |\hat{f}(\xi)|^2.$$

One can show that $P_V f(u,\xi) = |f(u)|^2 |\hat{f}(\xi)|^2$ forces f to be a Gaussian function. Furthermore Gaussians are the only family of functions whose W-V distribution remains positive.

One of the most important properties of the W-V distribution is its unitarity. This would not be possible if the transform remained positive.

Theorem 4.6 'Moyal'
Let f and g in $L^2(\mathbb{R})$, then

$$2\pi \left| \int_{-\infty}^{\infty} f(t)\overline{g(t)}\ \mathrm{d}t \right|^2 = \iint P_V f(u,\xi) P_V g(u,\xi)\ \mathrm{d}u\ \mathrm{d}\xi. \tag{4.9}$$

Proof: Let us compute the right hand side

$$\begin{aligned} I &= \iint P_V f(u,\xi) P_V g(u,\xi)\ \mathrm{d}u\ \mathrm{d}\xi \\ &= \iiiint f\left(u+\frac{\tau}{2}\right) f^*\left(u-\frac{\tau}{2}\right) g\left(u+\frac{\tau'}{2}\right) g^*\left(u-\frac{\tau'}{2}\right) e^{-i\xi(\tau+\tau')}\ \mathrm{d}\tau\ \mathrm{d}\tau'\ \mathrm{d}u\ \mathrm{d}\xi. \end{aligned}$$

The integral

$$\int_{-\infty}^{\infty} e^{-i\xi(\tau+\tau')} = 2\pi\delta(\tau+\tau')$$

is the Fourier transform of the function $h(t) \equiv 1$ at point $\tau+\tau'$. Inserting this result yields

$$\begin{aligned} I &= 2\pi \iiint f\left(u+\frac{\tau}{2}\right) \overline{f\left(u-\frac{\tau}{2}\right)} g\left(u+\frac{\tau'}{2}\right) \overline{g\left(u-\frac{\tau'}{2}\right)} \delta(\tau+\tau')\ \mathrm{d}\tau\ \mathrm{d}\tau'\ \mathrm{d}u \\ &= 2\pi \iint f\left(u+\frac{\tau}{2}\right) \overline{f\left(u-\frac{\tau}{2}\right)} g\left(u+\frac{\tau'}{2}\right) \overline{g\left(u-\frac{\tau'}{2}\right)}\ \mathrm{d}\tau\ \mathrm{d}u. \end{aligned}$$

A change of variables

$$\left.\begin{aligned} t(u,\tau) &= u+\tfrac{\tau}{2} \\ s(u,\tau) &= u-\tfrac{\tau}{2} \end{aligned}\right\} \quad \left|\frac{\partial(s,t)}{\partial(u,\tau)}\right| = 1$$

completes the proof. □

Summary 4.7 Properties of the Wigner-Ville distribution

Function	**Wigner-Ville**
$t \to f(t)$	$(u,\xi) \to P_V f(u,\xi)$
$e^{i\phi} f(t)$	$P_V f(u,\xi)$
$af(t)$	$\|a\|^2 P_V f(u,\xi)$
$f(t-u_0)$	$P_V f(u-u_0,\xi)$
$e^{i\xi_0 t} f(t)$	$P_V f(u,\xi-\xi_0)$
$e^{iat^2} f(t)$	$P_V f(u,\xi-2au)$
$\frac{1}{\sqrt{s}} f\left(\frac{t}{s}\right)$	$P_V f\left(\frac{u}{s}, s\xi\right)$

Example 4.8 Let g be a symmetric and sufficient smooth window function (i.e. $g(t) = 0$ for all $|t| > T$). Its W-V distribution $P_V g(u, \xi)$ is centered at $u = \xi = 0$. The W-V distribution of the time-frequency atom

$$f(t) = ae^{i\phi_0} g(t - u_0) e^{i\xi_0 t}$$

is derived from $P_V g(u, \xi)$ by applying the rules of summary 4.7

$$P_V f(u, \xi) = |a|^2 P_V g(u - u_0, \xi - \xi_0).$$

Its energy is concentrated in the neighbourhood of (u_0, ξ_0).

4.2 Interferences and Positivity of the Wigner-Ville distribution

The Wigner-Ville distribution is lacking in two important aspects. One is the already mentioned nonpositivity and the other results from so called interference phenomenons created by **cross terms**.

Let $f = f_1 + f_2$ be a composite signal. Since the Wigner-Ville distribution is a quadratic form,

$$P_V f = P_V f_1 + P_V f_2 + P_V[f_1, f_2] + P_V[f_2, f_1],$$

where $P_V[h, g]$ is the **cross Wigner-Ville distribution** of two signals

$$P_V[h, g](u, \xi) = \int_{-\infty}^{\infty} h\left(u + \frac{\tau}{2}\right) g^*\left(u - \frac{\tau}{2}\right) e^{-i\tau\xi} \, \mathrm{d}\tau.$$

The interference term

$$I[f_1, f_2] := P_V[f_1, f_2] + P_V[f_2, f_1]$$

is a real function but the term may be nonzero at points (u, ξ) where neither f_1, f_2 nor their Fourier transforms are localized.

Example 4.9 Let us consider the two time-frequency atoms defined by

$$\begin{aligned} f_1(t) &= a_1 e^{i\phi_1} g(t - u_1) e^{i\xi_1 t}, \\ f_2(t) &= a_2 e^{i\phi_2} g(t - u_2) e^{i\xi_2 t}, \end{aligned}$$

where g is a time window centered at $t = 0$. Their Wigner-Ville distributions are

$$\begin{aligned} P_V f_1(u, \xi) &= a_1^2 P_V g(u - u_1, \xi - \xi_1), \\ P_V f_2(u, \xi) &= a_2^2 P_V g(u - u_2, \xi - \xi_2). \end{aligned}$$

The interference term is given by

$$I[f_1, f_2](u, \xi) = 2a_1 a_2 P_V g(u - u_0, \xi - \xi_0) \cos((u - u_0)\triangle\xi - (\xi - \xi_0)\triangle u + \triangle\phi)$$

with

$$u_0 = \frac{u_1 + u_2}{2}, \ \xi_0 = \frac{\xi_1 + \xi_2}{2},$$

$$\triangle u = u_1 - u_2, \ \triangle\xi = \xi_1 - \xi_2 \quad \text{and} \quad \triangle\phi = \phi_1 - \phi_2 + u_0\triangle\xi.$$

It is an oscillatory waveform centered at (u_0, ξ_0) but neither f_1 nor f_2 are concentrated at this point.

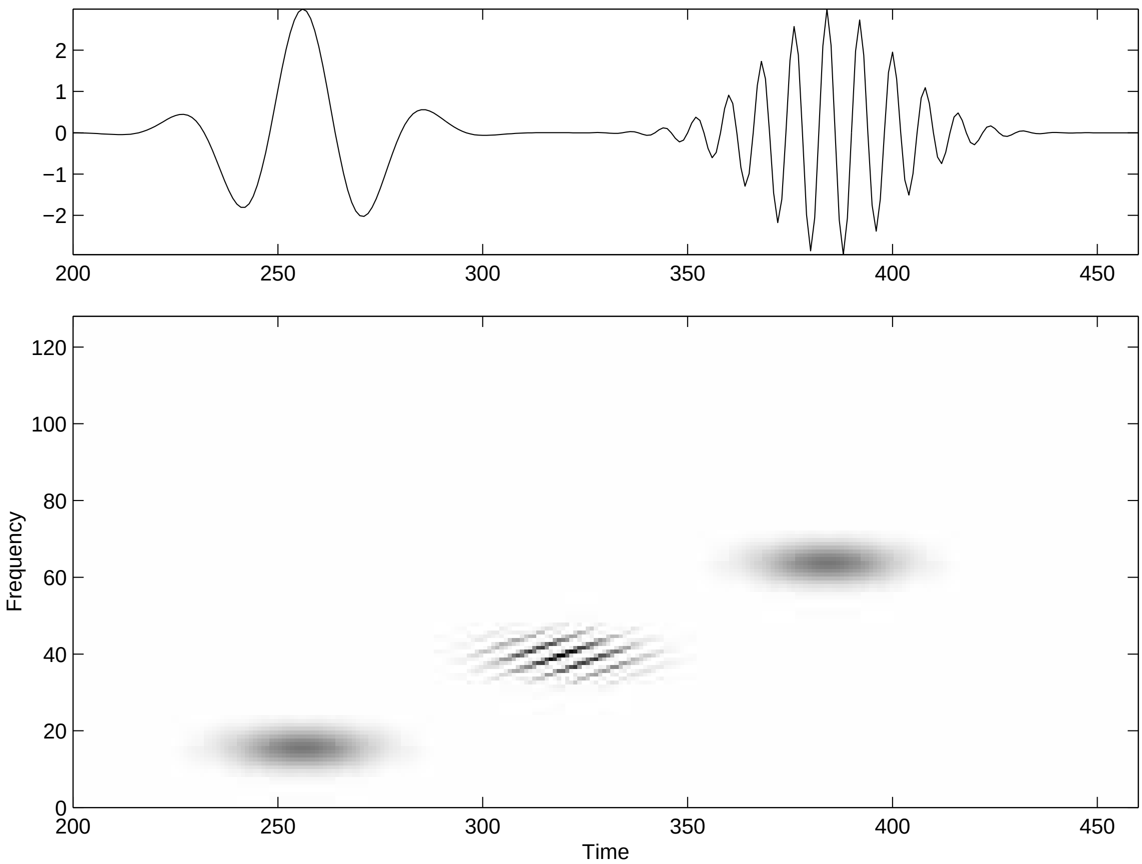


Figure 4.1: Wigner-Ville distribution $P_V f(u,\xi)$ of two Gabor atoms shown at the top. The oscillating interferences are centered at the middle time-frequency location.

Positive time-frequency distributions totally remove the inteference terms but produce a loss of resolution. The following theorem shows that there exists no positive time-frequency distribution which satisfy (4.4) and (4.5). This is one of the reasons why the Wigner-Ville distribution - despite its disadvantages - is a widely used tool in practice.

Theorem 4.10 *There is no positive quadratic energy distribution Pf that satisfies the following time and frequency marginal integrals:*

$$\int_{-\infty}^{\infty} Pf(u,\xi)\ \mathrm{d}\xi = 2\pi|f(u)|^2, \quad \int_{-\infty}^{\infty} Pf(u,\xi)\ \mathrm{d}u = |\hat{f}(\xi)|^2.$$

Proof: Suppose Pf to be a positive quadratic distribution that satisfies these marginals. Since $Pf(u,\xi) \geq 0$, this implies that if the support of f is included in an interval I then $Pf(u,\xi) = 0$ for $u \notin I$. We can associate to the quadratic form Pf a bilinear distribution defined for any f and g by

$$P[f,g] = \frac{1}{4}(P(f+g) - P(f-g)).$$

Let f_1 and f_2 be two non-zero signals whose supports are two intervals I_1 and I_2 that do not intersect, so that $f_1 f_2 = 0$. Let $f = af_1 + bf_2$:

$$Pf = |a|^2 Pf_1 + a\bar{b}P[f_1, f_2] + \bar{a}bP[f_2, f_1] + |b|^2 Pf_2.$$

Since I_1 does not intersect I_2, $Pf_1(u, \xi) = 0$ for $u \in I_2$. The positivity of Pf for all $a, b \in \mathbb{C}$ forces

$$P[f_1, f_2](u, \xi) = P[f_2, f_1](u, \xi) = 0$$

for all $u \in I_2$. Similarly we can show that these cross terms are zero for $u \in I_1$ and hence

$$Pf(u, \xi) = |a|^2 Pf_1(u, \xi) + |b|^2 Pf_2(u, \xi).$$

Applying $\int_{-\infty}^{\infty} \, \mathrm{d}\xi$ on both sides and using the marginal integrals yields

$$|\hat{f}(\xi)|^2 = |a|^2 |\hat{f}_1(\xi)|^2 + |b|^2 |\hat{f}_2(\xi)|^2.$$

Since $\hat{f}(\xi) = a\hat{f}_1(\xi) + b\hat{f}_2(\xi)$ it follows that $\hat{f}_1(\xi)\hat{f}_2(\xi) = 0$. We can assume w.l.o.g. that $\hat{f}_1$ is zero on a compact interval. But this is a contradiction to Theorem 1.10. $\square$

5 Wavelet Bases

Now, we are looking for wavelets $\Psi \in L^2(\mathbb{R})$ such that the family of scaled and shifted wavelets,

$$\boxed{\left\{ \Psi_{j,k} := 2^{j/2} \Psi \left(\frac{t - 2^{-j}k}{2^{-j}} \right) \right\}_{(j,k) \in \mathbb{Z}^2}} \tag{5.1}$$

is a basis of $L^2(\mathbb{R})$. Of course, the optimal case is obtained if (5.1) forms an orthonormal basis.

5.1 Frames and Riesz Bases

Let us consider a Hilbert space H with induced norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ and a family of vectors

$$\{\phi_k : k \in J\} \subseteq H$$

where J is an approbiate countable index set. A natural question is, given an arbitrary vector $f \in H$, can f be rediscovered from the quantities $\langle f, \phi_k \rangle$, $k \in J$? Or, equivalently, can one invert the operator $U : H \to \operatorname{Im} U$ given by

$$(Uf)[k] := \langle f, \phi_k \rangle \ ?$$

Indeed this is possible if $\{\phi_k : k \in J\}$ forms a frame.

Definition 5.1 *A family of vectors $\{\phi_k : k \in J\} \subseteq H$ is called a* **frame** *if there exist frame constants $A, B > 0$ such that for all $f \in H$*

$$A \sum_{k \in J} |\langle f, \phi_k \rangle|^2 \leq \|f\|^2 \leq B \sum_{k \in J} |\langle f, \phi_k \rangle|^2. \tag{5.2}$$

A frame is called a **tight frame** *if $A = B$.*

Note that the vectors $\phi_k,\ k \in J$ are not necessarily linearly independent.

Example 5.2 $H = \mathbb{R}^2 = \text{span}\{\vec{e}_1, \vec{e}_2\}$
Consider the three vectors

$$\phi_1 = \vec{e}_1, \quad \phi_2 = -\frac{\vec{e}_1}{2} + \frac{\sqrt{3}}{2}\vec{e}_2 \quad \text{and} \quad \phi_3 = -\frac{\vec{e}_1}{2} - \frac{\sqrt{3}}{2}\vec{e}_2.$$

The angle between ϕ_i and ϕ_j is $\frac{2\pi}{3}$ for $i \neq j$. It can be shown that $\{\phi_1, \phi_2, \phi_3\}$ forms a tight frame with $A = B = \frac{3}{2}$.

Theorem 5.3 *The operator $U : H \to l^2(J)$ defined by $(Uf)[k] := \langle f, \phi_k \rangle$ is a bounded and invertible operator $H \to \text{Im}\, U$. $\text{Im}\, U$ is a closed subspace of $l^2(J)$.*

Proof: We only give an outline of the proof. For more details see [1].
The adjoint of U maps $l^2(J) \to H$ and is given by

$$(U^\star a[\cdot]) = \sum_{k\in J} a[k]\phi_k.$$

We get $U^\star U f = U^\star[\langle f, \phi_\cdot\rangle] = \sum_{k\in J}\langle f, \phi_k\rangle \phi_k$, $U^\star U$ is a symmetric operator. One can show that $U^\star U : H \to H$ is bounded and invertible.

Let $f \in H$ be arbitrarily,

$$\sum_{k\in J} |\langle f, \phi_k\rangle|^2 = \langle U^\star U f, f\rangle = \langle Uf, Uf\rangle = \|Uf\|^2.$$

Together with (5.2) we obtain the inequality

$$A\|Uf\|^2 \leq \|f\|^2 \leq B\|Uf\|^2.$$

U is bounded since $\|Uf\|^2 \leq \frac{1}{A}\|f\|^2$. Furthermore from $\|f\|^2 \leq B\|Uf\|^2$ we get the injectivity of U and the existence of a bounded inverse $U^{-1} : \text{Im}\, U \to H$. Finally the closed graph theorem tells us that $\text{Im}\, U$ is a closed subspace of $l^2(J)$. □

Definition 5.4 *A family $\{\phi_k : k \in J\} \subseteq H$ is called a* **Riesz basis** *if every f posseses a unique representation*

$$f = \sum_{k\in J} a[k]\phi_k$$

with approbiate coefficients $a[k]$ and if there exist constants $C_1, C_2 > 0$ such that

$$\boxed{C_1 \sum_{k\in J} |a[k]|^2 \leq \|f\|^2 \leq C_2 \sum_{k\in J} |a[k]|^2.} \tag{5.3}$$

5.2 Multi Resolution Analysis (MRA)

Definition 5.5 *A family of closed linear subspaces* $V_j \subseteq L^2(\mathbb{R}),\ j \in \mathbb{Z}$ *is a* **multiresolution approximation** *if the following 6 properties are satisfied:*

1. $f \in V_j \ \Leftrightarrow\ f(\cdot - 2^{-j}k) \in V_j)$ *for all* $(j,k) \in \mathbb{Z}^2$,
2. $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$,
3. $f \in V_j \ \Leftrightarrow\ f(2\cdot) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$,
4. $\lim\limits_{j\to-\infty} V_j = \bigcap\limits_{j=-\infty}^{\infty} V_j = \{0\}$,
5. $\lim\limits_{j\to\infty} V_j = \overline{\bigcup\limits_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R})$,
6. *there exists a function* $\varphi \in L^2(\mathbb{R})$ *such that* $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ *is a Riesz basis of* V_0.

Remark 5.6

(a) The function φ in Property 6 is called a **scaling function** or **generating function**.

(b) Property 6 can be replaced by the existence of a finite family of scaling functions $\{\varphi_1, \dots, \varphi_N\}$. This leads to the concept of **Multiwavelets**.

Alternatively we can give a constructive description. Let us consider a **scaling function** $\varphi \in L^2(\mathbb{R})$ and

$$\varphi_{j,k} = 2^{j/2}\varphi(2^j(\cdot - k)),\ j,k \in \mathbb{Z},$$
$$V_j = \text{cl span}\{\varphi_{j,k},\ k \in \mathbb{Z}\}.$$

φ should satisfy the following three relations:

1. $\sum\limits_{k\in\mathbb{Z}} \varphi(x-k) = 1$
2. **Refinement relation**:
   There exist $a[k],\ k \in \mathbb{Z}$ such that
   $$\varphi(x) = \sqrt{2}\sum_{k\in Z} a[k]\varphi(2x-k), \quad \text{i.e.,} \quad \varphi_{0,0} = \sum_{k\in\mathbb{Z}} a[k]\varphi_{1,k}.$$
3. $\{\varphi_{0,k} :\ k \in \mathbb{Z}\}$ forms a Riesz basis of V_0.

Note that the third relation also implies that every $\{\varphi_{j,k} : k \in \mathbb{Z}\}$ is a Riesz basis of V_j with certain constants $C_1, C_2 > 0$.

Construction of Wavelet Bases with MRA

If we take an MRA we know that V_j is included in V_{j+1}. We can find a subspace $W_j \subset L^2(\mathbb{R})$ which is the complement of V_j in V_{j+1} and satisfies

$$V_j \cap W_j = \{0\}, \quad V_{j+1} = V_j \oplus W_j \quad \text{and} \quad \cos(V_j, W_j) > 0.$$

The last condition is equivalent to

(a) "sharpened Cauchy inequality"
There exists a γ with $0 < \gamma < 1$ such that

$$|\langle v_j, w_j \rangle| \leq \gamma \|v_j\| \|w_j\|, \quad \forall v_j \in V_j, \ w_j \in W_j.$$

(b) "inverse triangle inequality"
There exists a $c > 0$ such that

$$\|v_j\|^2 + \|w_j\|^2 \leq c\|v_j + w_j\|^2, \quad \forall v_j \in V_j, \ w_j \in W_j.$$

If W_j is the orthogonal complement we can replace this by the equality $\|v_j\|^2 + \|w_j\|^2 = \|v_j + w_j\|^2$.

Again the subspaces W_j are invariant under translation, i.e.,

$$f \in W_j \ \Leftrightarrow \ f(\cdot - 2^{-j}k) \in W_j, \quad \forall k \in \mathbb{Z}.$$

To be more specific: There exists a **wavelet** (the so called mother wavelet) $\Psi \in L^2(\mathbb{R})$ such that

$$\{\Psi_{j,k}(x) := 2^{j/2}\Psi(2^j t - k)(x), \ k \in \mathbb{Z}\}$$

forms a Riesz basis of W_j. Note that we are now very close to the desired result we drawed at the beginning of this section (see (5.1)). Using the decomposition

$$V_{j+1} = W_j \oplus V_j = V_0 \oplus \bigoplus_{l=0}^{j} W_l = \bigoplus_{l=-\infty}^{j} W_l$$

we obtain with $j \to \infty$ that $\bigoplus\limits_{l=-\infty}^{\infty} W_l = L^2(\mathbb{R})$. In particular this means that

$$\left\{ \Psi_{j,k} := 2^{j/2}\Psi\left(\frac{t - 2^{-j}k}{2^{-j}}\right) \right\}_{(j,k)\in\mathbb{Z}^2}$$

is a basis of $L^2(\mathbb{R})$. We call it a **wavelet basis** if (5.1) is even a Riesz basis. **Orthonormal wavelet bases** satisfy

$$\boxed{\langle \Psi_{j,k}, \Psi_{j,k'} \rangle = \delta_{k,k'} \quad \forall j \in \mathbb{Z}.}$$

Example 5.7 Piecewise constant approximation
Let us consider the **box function**

$$\theta^1(x) = \chi_{[0,1)}(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 0 & \text{otherwise,} \end{cases}$$

then $\varphi(x) := \theta^1(x)$ is a scaling function. Indeed $\{\varphi_{0,k} = \varphi(\cdot - k),\ k \in \mathbb{Z}\}$ forms an orthonormal basis of V_0. The space V_j consists of all piecewise constant functions, subordinated to a uniform mesh width $h_j = 2^{-j}$. I.e., all functions $f_j \in V_j$ are constant on the intervals $[2^{-j}k, 2^{-j}(k+1))$.

The refinement relation is

$$\varphi(x) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\varphi(2x) + \frac{1}{\sqrt{2}}\varphi(2x-1)\right) = \frac{1}{\sqrt{2}}\varphi_{1,0}(x) + \frac{1}{\sqrt{2}}\varphi_{1,1}(x).$$

A matching Wavelet may be found by

$$\Psi(x) = \varphi(2x) - \varphi(2x-1) = \frac{1}{\sqrt{2}}(\varphi_{1,0}(x) - \varphi_{1,1}(x)).$$

So the following equalities are satisfied:

$$\begin{aligned}
\langle \Psi, \varphi \rangle &= \int_0^1 \Psi(x)\ \mathrm{d}x = \int_0^{1/2}\ \mathrm{d}x - \int_{1/2}^1\ \mathrm{d}x = 0 \\
\langle \Psi(\cdot - k), \Psi(\cdot - k') \rangle &= \delta_{k,k'} \\
\langle \Psi, \Psi(2\cdot) \rangle &= 0 \\
\langle \Psi_{j,k}, \Psi_{j',k'} \rangle &= \delta_{j,j'}\delta_{k,k'} \\
\langle \Psi_{j,k}, \varphi_{l,k'} \rangle &= 0, \quad \forall l \le j,\ \forall k,k' \in \mathbb{Z}\ \Rightarrow\ W_j \perp V_j, V_{j-1}, \ldots
\end{aligned}$$

The Wavelet basis defined by those $\Psi_{j,k}$ is called the **Haar basis**. It has the drawback that the functions of V_j are not smooth enough, indeed they are not even continuous. This implies low regularity and slow approximation of smooth functions.

Example 5.8 Shannon approximation
Here we consider the scaling function

$$\varphi(x) := 2\pi\hat{\theta}_0(x) = \frac{\sin \pi x}{\pi x}.$$

Then $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ forms again an orthonormal basis of V_0.

Example 5.9 B-spline approximation
Let us consider

$$\theta^d(x) = (\underbrace{\theta^1 * \cdots * \theta^1}_{d \text{ times}})(x),$$

then

$$\hat{\theta}^d(\xi) = (\hat{\theta}^0(\xi))^m = \left(\frac{\sin \xi/2}{\xi/2}\right)^d e^{-i\xi/2}.$$

These are piecewise polynomials of degree $d-1$ on intervals $(2^{-j}k, 2^{-j}(k+1)),\ k \in \mathbb{Z}$. They are $d-2$ times continuously differentiable.

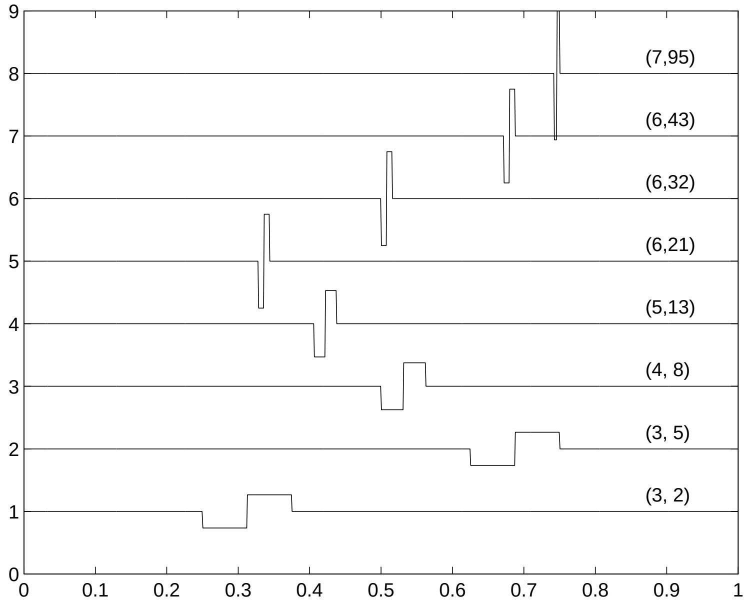


Figure 5.1: Some Haarlets at various scales and locations.

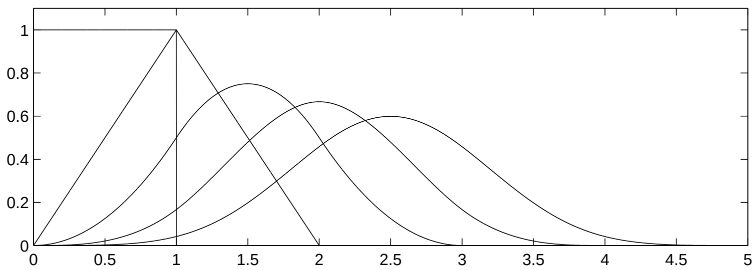


Figure 5.2: B-splines of order $0, \ldots, 4$.

The following lemma gives an answer to the question whether an $L^2(\mathbb{R})$-function φ generates with $\{\varphi(\cdot - k),\ k \in \mathbb{Z}\}$ a Riesz basis of V_0.

Lemma 5.10 *A family $\{\varphi(\cdot - k),\ k \in \mathbb{Z}\}$ forms a Riesz basis in $L^2(\mathbb{R})$ if and only if there exist constants $A, B > 0$ such that*

$$\boxed{A \leq \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi - 2k\pi)| \leq B < \infty} \tag{5.4}$$

for all $\xi \in [-\pi, \pi]$.

Proof: Let $f \in V_0$ be decomposed as

$$f(t) = \sum_{k=-\infty}^{\infty} a[k]\varphi(t-k) \tag{5.5}$$

with approbiate coefficients $a[k]$. Taking the Fourier transform of (5.5) yields $\hat{f}(\xi) = \tilde{a}(\xi)\hat{\varphi}(\xi)$ where

$$\tilde{a}(\xi) = \sum_{k=-\infty}^{\infty} a[k]e^{-ik\xi}.$$

Then

$$\begin{aligned} \|f\|^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{a}(\xi)|^2|\hat{\varphi}(\xi)|^2 \, d\xi \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} |a[k]|^2 \int_0^{2\pi} |\hat{\varphi}(\xi+2k\pi)|^2 \, d\xi. \end{aligned} \tag{5.6}$$

On the other hand the family $\{\varphi(\cdot - k),\ k \in \mathbb{Z}\}$ forms a Riesy basis iff

$$A\sum_{k=-\infty}^{\infty} |a[k]|^2 \le \|f\|^2 \le B\sum_{k=-\infty}^{\infty} |a[k]|^2. \tag{5.7}$$

By the discrete Plancherel formula we get

$$\sum_{k=-\infty}^{\infty} |a[k]|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\tilde{a}(\xi)|^2 \, d\xi.$$

Together with (5.6) we conclude that

$$\|f\|^2 \ge \|a[\cdot]\|_{l^2(\mathbb{Z})} \inf_{\xi\in[0,2\pi]} \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi - 2k\pi)|^2$$

and

$$\|f\|^2 \le \|a[\cdot]\|_{l^2(\mathbb{Z})} \sup_{\xi\in[0,2\pi]} \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi - 2k\pi)|^2.$$

which shows (5.7). The functions $\varphi(\cdot - k)$ are linearly independent since $f \equiv 0$ implies $a[k] = 0$ for all $k \in \mathbb{Z}$ and vice versa.

In order to prove that the condition is necessary let us assume there exists a $\xi_0 \in [0, 2\pi]$ such that

$$\sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi_0 + 2k\pi)|^2 = 0.$$

We construct a sequence $(\tilde{a}_n(\xi))_{n=1}^{\infty}$ such that

$$\text{supp } \tilde{a}_n(\xi) \subset \left[\xi_0 - \frac{1}{n}, \xi_0 + \frac{1}{n}\right]$$

and $\|\tilde{a}(\xi)\|_{L^2([0,2\pi])} = 1$. Then we get

$$\tilde{a}_n \cdot \hat{\varphi}(\xi) \to 0 \quad \text{for} \quad n \to \infty,$$

which contradicts the Riesz basis property. □

5.3 Orthogonal Wavelet Bases

Theorem 5.11 *Let $\theta \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\{\theta(\cdot - k) \ : \ k \in \mathbb{Z}\}$ be a Riesz basis in V_0. Then φ defined by*

$$\hat{\varphi}(\omega) = \frac{1}{\left(\sum_{k=-\infty}^{\infty} |\hat{\theta}(\omega + 2k\pi)|^2\right)^{1/2}} \hat{\theta}(\omega) \tag{5.8}$$

generates

$$\{\varphi_{j,k}(t) \ : \ 2^{j/2}\varphi(2^j t - k), \ k \in \mathbb{Z}\}$$

as an orthonormal basis of V_j.

Proof: We represent

$$\varphi(t) = \sum_{k=-\infty}^{\infty} a[k]\varphi(t-k)$$

or, equivalently,

$$\hat{\varphi}(\omega) = \hat{a}(\omega)\hat{\theta}(\omega),$$

where $\hat{a}(\omega) = \sum_{k=-\infty}^{\infty} a[k]e^{-ik\omega}$ is 2π-periodic. The orthonormality condition is

$$\begin{aligned} \langle \varphi(\cdot - k), \varphi(\cdot - k')\rangle &= \int_{-\infty}^{\infty} \varphi(t-k)\varphi^*(t-k') \text{ d}t \\ &= (\varphi * \tilde{\varphi})(k' - k) = \delta_{k,k'}, \end{aligned}$$

where $\tilde{\varphi}(t) = \varphi^*(-t)$ The Fourier transform of this equality gives

$$\widehat{\varphi * \tilde{\varphi}} = \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1. \tag{5.9}$$

(5.8) is verified by choosing

$$\hat{a}(\omega) = \frac{1}{\left(\sum_{k=-\infty}^{\infty} |\hat{\theta}(\omega + 2k\pi)|^2\right)^{1/2}}.$$

□

Example 5.12 For piecewise constant box functions and the Shannon scaling function we have constructed orthonormal bases in Examples 5.7 and 5.8. However, the B-spline bases are not orthogonal for $d > 1$. Therefore we insert the known Fourier transform

$$\hat{\theta}(\omega) = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^d e^{i\frac{\omega}{2}\varepsilon},$$

where $\varepsilon = 1$ if d is odd and $\varepsilon = 0$ for even d, in the previous theorem and obtain

$$\hat{\varphi}(\omega) = \frac{e^{-i\frac{\omega}{2}\varepsilon}}{\omega^d\sqrt{S_{2d}(\omega)}}, \quad \text{with} \quad S_{2d}(\omega) = \sum_{k=-\infty}^{\infty} (\omega + 2k\pi)^{-2d}.$$

The sum $S_{2d}(\omega)$ can be computed explicitly. At first we observe by differentation that

$$S_{2d}(\omega) = S''_{2(d-1)}(\omega) = \cdots = S_2^{2d-2}(\omega).$$

Together with the known formula

$$S_2(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{\omega + 2k\pi} = \frac{1}{4\sin^2(\frac{\omega}{2})}$$

we obtain for linear splines ($d = 2$)

$$S_4(\omega) = \frac{1 + 2\cos^2\frac{\omega}{2}}{48\sin^4\frac{\omega}{2}} \quad \text{and} \quad \hat{\varphi}(\omega) = \frac{4\sqrt{3}\sin^2\frac{\omega}{2}}{\omega^2\sqrt{1 + 2\cos^2\frac{\omega}{2}}}.$$

and cubic splines ($d = 4$)

$$S_8(\omega) = \frac{5 + 30\cos^2\frac{\omega}{2} + 30\sin^2\frac{\omega}{2}\cos^2\frac{\omega}{2} + 70\cos^4\frac{\omega}{2} + 2\sin^4\frac{\omega}{2}\cos^4\frac{\omega}{2} + \frac{2}{3}\sin^6\frac{\omega}{2}}{105 \cdot 2^8\sin^8\frac{\omega}{2}}.$$

The corresponding cubic spline scaling function is displayed in Figure 5.3.

Conjugate Mirror Filters

Let us consider the orthogonal projection $P_j : L^2(\mathbb{R}) \to V_j$ given by

$$P_j f(t) := \sum_{k=-\infty}^{\infty} \langle f, \varphi_{j,k}\rangle \varphi_{j,k}(t) = \sum_{k=-\infty}^{\infty} 2^j \int_{-\infty}^{\infty} f(x)\overline{\varphi(2^j x - k)}\,\mathrm{d}x\ \varphi(2^j t - k).$$

In the case of nested subspaces $\cdots \subset V_j \subset V_{j+1} \subset \ldots,\ j \in \mathbb{Z}$ we have

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\} \qquad \Leftrightarrow \qquad \lim_{j\to\infty} P_j f(t) = 0,\ \forall f \in L^2(\mathbb{R}), t \in \mathbb{R}$$

where the limit is taken pointwise. Furthermore we get the equivalence

$$\bigcup_{j\in\mathbb{Z}} V_j \text{ is dense in } L^2(\mathbb{R}) \qquad \Leftrightarrow \qquad \lim_{j\to\infty} \|f - P_j f\| = 0,\ \forall f \in L^2(\mathbb{R}).$$

The inclusion $V_j \subset V_{j+1}$ implies the existence of coefficients $m_{q,k}^{j,0}$ such that

$$\varphi_{j,k} = \sum_{q=-\infty}^{\infty} m_{q,k}^{j,0}\ \varphi_{j+1,k}.$$

This can be reduced to the case $j = 0$ by dilation and translation , i.e. to the relation $V_0 \subset V_1$:

$$\varphi(x) = \varphi_{0,0}(x) = \sum_{q=-\infty}^{\infty} m_{q,0}^{0,0}\ \varphi_{1,q}(x) = \sqrt{2}\sum_{q=-\infty}^{\infty} h[q]\varphi(2x - q), \tag{5.10}$$

or

$$\frac{1}{\sqrt{2}}\varphi\left(\frac{x}{2}\right) = \sum_{q=-\infty}^{\infty} h[q]\varphi(x-q),$$

with

$$h[q] = \langle \varphi, \varphi_{1,q}\rangle = \sqrt{2}\int_{-\infty}^{\infty} \varphi(x)\overline{\varphi(2x-q)}\,\mathrm{d}x.$$

The matrix M with $[M]_{q,k} = m_{q,k}^{0,0}$ is called the **refinement mask matrix** and $q \to h[q]$ the mask or filter. If φ has compact support then only finite coefficients $h[k]$ are nonzero and $\hat{h}$ is a trigonometric polynom. In that case we call h a **conjugate mirror filter**.

Taking the Fourier transform of (5.10) yields

$$\hat{\varphi}(2\omega) = \frac{1}{\sqrt{2}}\hat{h}(\omega)\hat{\varphi}(\omega) \tag{5.11}$$

for $\hat{h}(\omega) = \sum_{k=-\infty}^{\infty} h[k]e^{-ik\omega}$. For any $j \geq 0$ the equality (5.11) implies

$$\hat{\varphi}(2^{-j+1}\omega) = \frac{1}{\sqrt{2}}\hat{h}(2^{-j}\omega)\hat{\varphi}(2^{-j}\omega),$$

and by substitution

$$\hat{\varphi}(\omega) = \left(\prod_{k=1}^{j}\frac{\hat{h}(2^{-k}\omega)}{\sqrt{2}}\right)\hat{\varphi}(2^{-j}\omega).$$

If $\hat{\varphi}$ is continuous at $\omega = 0$ then $\lim_{j\to\infty}\hat{\varphi}(2^{-j}\omega) = \hat{\varphi}(0)$ and so

$$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty}\frac{\hat{h}(2^{-k}\omega)}{\sqrt{2}}\hat{\varphi}(0). \tag{5.12}$$

This means that the scaling function is completely determined by the values of $h[k]$ for $k \in \mathbb{Z}$ which are the filter coefficients. The following theorem gives necessary and sufficient conditions on $\hat{h}$ to guarantee that the infinite product (5.12) is the Fourier transform of a scaling function.

Theorem 5.13 (Mallat, Meyer 89)

Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a scaling function such that $\langle\varphi(\cdot), \varphi(\cdot - k)\rangle = \delta_{0,k}$. Then the Fourier series $\hat{h}(\omega) = \sum_{k=-\infty}^{\infty} h[k]e^{-ik\omega}$, where $h[k] = \langle\frac{1}{\sqrt{2}}\varphi(\frac{\cdot}{2}), \varphi(\cdot - k)\rangle$, satisfies

$$\hat{h}(0) = \sqrt{2}, \tag{5.13}$$

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega+\pi)|^2 = 2, \quad \text{for all } \omega \in \mathbb{R}. \tag{5.14}$$

Conversely, if a 2π-periodic function $\hat{h}$ satisfies (5.13, 5.14) and $\hat{h} \in C^1([-\varepsilon, \varepsilon])$ for some $\varepsilon > 0$ and if

$$\inf_{\omega\in[-\frac{\pi}{2},\frac{\pi}{2}]} |\hat{h}(\omega)| > 0 \tag{5.15}$$

then

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} \frac{\hat{h}(2^{-j}\omega)}{\sqrt{2}} \tag{5.16}$$

is the Fourier transform of some scaling function φ *such that* $\langle \varphi(\cdot), \varphi(\cdot - k)\rangle = \delta_{0,k}$*. Additionally,* φ *generates an MRA by* $V_j = \overline{span\{\varphi_{j,k}, k \in \mathbb{Z}\}}$*.*

Proof: This theorem is a central result with a long and technical proof. Therefore it is divided in several parts.

Proof of the necessary condition (5.14) The orthogonality $\langle \varphi(\cdot), \varphi(\cdot - k)\rangle = \delta_{0,k}$ implies that (see (5.9) in the proof of theorem 5.11)

$$\sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1, \quad \forall \omega \in \mathbb{R}.$$

Inserting $\hat{\varphi}(\omega) = \frac{1}{\sqrt{2}}\hat{h}(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2})$ (see (5.11)) yields

$$\sum_{k=-\infty}^{\infty} \left|\hat{h}\left(\frac{\omega}{2} + k\pi\right)\right|^2 \left|\hat{\varphi}\left(\frac{\omega}{2} + k\pi\right)\right|^2 = 2.$$

Exploiting the 2π-periodicity of $\hat{h}$, we can seperate the even and odd integer terms of the above sum

$$\left|\hat{h}\left(\frac{\omega}{2}\right)\right|^2 \sum_{k=-\infty}^{\infty} \left|\hat{\varphi}\left(\frac{\omega}{2} + 2k\pi\right)\right|^2 + \left|\hat{h}\left(\frac{\omega}{2} + \pi\right)\right|^2 \sum_{k=-\infty}^{\infty} \left|\hat{\varphi}\left(\frac{\omega}{2} + \pi + 2k\pi\right)\right|^2 = 2.$$

Again, we observe that

$$\sum_{k=-\infty}^{\infty} \left|\hat{\varphi}\left(\frac{\omega}{2} + 2k\pi\right)\right|^2 = \sum_{k=-\infty}^{\infty} \left|\hat{\varphi}\left(\frac{\omega}{2} + \pi + 2k\pi\right)\right|^2 = 1.$$

Setting $\xi = \frac{\omega}{2}$ yields

$$|\hat{h}(\xi)|^2 + |\hat{h}(\xi + \pi)|^2 = 2,$$

which proves (5.14).

Proof of the necessary condition (5.13) Let $P_j : L^2(\mathbb{R}) \to V_j$ be defined as

$$P_j u = \sum_{k=-\infty}^{\infty} \langle u, \varphi_{j,k}\rangle \varphi_{j,k}.$$

Obviously P_j is an orthogonal projection which means $P_j P_j = P_j$ and $\langle P_j u, w\rangle = \langle u, P_j w\rangle$ for all $u, w \in L^2(\mathbb{R})$. From $\hat{\varphi}(0) = \frac{1}{\sqrt{2}}\hat{h}(0)\hat{\varphi}(0)$ we may conclude $\hat{h}(0) = \sqrt{2}$ provided that $\hat{\varphi}(0) \neq 0$. Indeed that can be shown by the following arguments.

The density of $\bigcup_{j=-\infty}^{\infty} V_j$ together with the Plancherel formula imply that

$$\lim_{j\to\infty} \|u - P_j u\| = \lim_{j\to\infty} \frac{1}{2\pi} \|\hat{u} - \widehat{P_j u}\| = 0. \tag{5.17}$$

To compute the Fourier transform of $P_j u$ let us denote $\varphi_j(t) = 2^{j/2}\varphi(2^j t)$. Then

$$P_j u(t) = \sum_{k=-\infty}^{\infty} (u * \varphi_j^*)(2^{-j}k) \cdot \varphi_j(t - 2^{-j}k) = \left(\varphi_j * \sum_{k=-\infty}^{\infty} (u * \varphi_j^*)(2^{-j}k)\delta\right)(t - 2^{-j}k).$$

The Fourier transform of $(u * \varphi_j^*)(t)$ is

$$2^{-j/2}\hat{u}(\omega) \cdot \hat{\varphi}^*(2^{-j}\omega).$$

A uniform sampling has a periodized Fourier transform which we calculated in Proposition 2.2, and hence

$$\widehat{P_j u}(\omega) = \hat{\varphi}(2^{-j}\omega) \sum_{k=-\infty}^{\infty} \hat{u}(\omega - 2^{j+1}k\pi) \cdot \overline{\hat{\varphi}}(2^{-j}[\omega - 2^{j+1}k\pi]). \tag{5.18}$$

We choose u with supp $u \subseteq [-\pi, \pi]$. Then for $j > 0$ and $\omega \in [-\pi, \pi]$ we obtain

$$\widehat{P_j u}(\omega) = \hat{u}(\omega)|\hat{\varphi}(2^{-j}\omega)|^2. \tag{5.19}$$

The L_2 convergence (5.17) implies pointwise convergence almost everywhere:

$$\lim_{j\to\infty} \widehat{P_j u}(\omega) = \hat{u}(\omega).$$

It follows from (5.19) that $\lim_{j\to\infty} |\hat{\varphi}(2^j\omega)| = |\hat{\varphi}(0)| = 1$.

Proof that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is orthonormal This is the first step to prove that the function φ whose Fourier transform is given by (5.16) is a scaling function. Since (5.13) we know that $|\hat{h}(\omega)| \leq \sqrt{2}$ and this implies that the infinite product (5.15) converges and $|\hat{\varphi}(\omega)| \leq 1$. Using Parsevals formula the orthonormality relation reads as follows

$$\langle \varphi(\cdot), \varphi(\cdot - k)\rangle = \int_{-\infty}^{\infty} \varphi(t)\overline{\varphi(t-k)}\, \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\varphi}(\omega)|^2 e^{2k\pi i\omega}\, \mathrm{d}\omega.$$

Verifying orthonormality is thus equivalent to showing that

$$\int_{-\infty}^{\infty} |\hat{\varphi}(\omega)|^2 e^{2k\pi i\omega}\, \mathrm{d}\omega = 2\pi\delta[k].$$

Let us consider the function

$$\hat{\varphi}_n(\omega) = \prod_{p=1}^{n} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}} \chi_{[-2^n\pi, 2^n\pi]}(\omega)$$

and the integrals

$$I_n[k] = \int_{-\infty}^{\infty} |\hat{\varphi}_n(\omega)|^2 e^{2k\pi i\omega}\ \mathrm{d}\omega = \int_{-2^n\pi}^{2^n\pi} \prod_{p=1}^{n} \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{2k\pi i\omega}\ \mathrm{d}\omega.$$

First, let us show that $I_n[k] = 2\pi\delta[k]$ for all $n \geq 1$. To do this, we decompose $I_n[k]$ as

$$I_n[k] = \int_{-2^n\pi}^{0} \prod_{p=1}^{n} \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{2k\pi i\omega}\ \mathrm{d}\omega + \int_{0}^{2^n\pi} \prod_{p=1}^{n} \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{2k\pi i\omega}\ \mathrm{d}\omega. \tag{5.20}$$

We change the variable $\omega' = \omega + 2^n\pi$ in the first integral taking into account the periodicity of $\hat{h}$, i.e. when $p < n$ then $|\hat{h}(2^{-p}[\omega' - 2^n\pi])|^2 = |\hat{h}(2^{-p}\omega')|^2$. When $p = n$ the hypothesis (5.13) tells us that

$$|\hat{h}(2^{-n}[\omega' - 2^n\pi])|^2 + |\hat{h}(2^{-n}\omega')|^2 = 2.$$

For $n > 1$ we can add both integrals in (5.20) and obtain

$$\begin{aligned} I_n[k] &= \int_{0}^{2^n\pi} \left(1 - \frac{|\hat{h}(2^{-n}\omega')|^2}{2}\right) \prod_{p=1}^{n-1} \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{2k\pi i\omega} + \prod_{p=1}^{n} \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{2k\pi i\omega}\ \mathrm{d}\omega \\ &= \int_{0}^{2^n\pi} \prod_{p=1}^{n-1} \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{2k\pi i\omega}\ \mathrm{d}\omega \qquad (5.21) \\ &= \int_{-2^{n-1}\pi}^{2^{n-1}\pi} \prod_{p=1}^{n-1} \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{2k\pi i\omega}\ \mathrm{d}\omega = I_{n-1}[k]. \end{aligned}$$

By induction we get $I_n[k] = I_1[k]$. Writing (5.21) for $n = 1$ gives

$$I_1[k] = \int_{0}^{2\pi} e^{2k\pi i\omega}\ \mathrm{d}\omega = 2\pi\delta[k],$$

which proves that $I_n[k] = 2\pi\delta[k]$, for all $n \geq 1$.

Now, we prove that $\hat{\varphi} \in L^2(\mathbb{R})$. For $\omega \in \mathbb{R}$

$$\lim_{n\to\infty} |\hat{\varphi}_n(\omega)|^2 = \prod_{p=1}^{\infty} \frac{|\hat{h}(2^{-p}\omega)|^2}{2} = |\hat{\varphi}(\omega)|^2.$$

Fatous lemma shows that

$$\int_{-\infty}^{\infty} |\hat{\varphi}(\omega)|^2\ \mathrm{d}\omega \leq \lim_{n\to\infty} \int_{-\infty}^{\infty} |\hat{\varphi}_n(\omega)|^2\ \mathrm{d}\omega = 2\pi,$$

because $I_n[0] = 2\pi$ for $n \geq 1$. Since

$$|\hat{\varphi}(\omega)|^2 e^{2k\pi i\omega} = \lim_{n\to\infty} |\hat{\varphi}_n(\omega)|^2 e^{2k\pi i\omega},$$

we will verify that

$$\int_{-\infty}^{\infty} |\hat{\varphi}(\omega)|^2 e^{2k\pi i\omega}\, \mathrm{d}\omega = \lim_{n\to\infty} \int_{-\infty}^{\infty} |\hat{\varphi}_n(\omega)|^2 e^{2k\pi i\omega}\, \mathrm{d}\omega = I_n[k] = 2\pi\delta[k]$$

by applying the dominated convergence theorem. This requires the proof of an upper bound condition. In our case we will prove that there exists a constant $C > 0$ such that

$$\left| |\hat{\varphi}_n(\omega)|^2 e^{2k\pi i\omega} \right| = |\hat{\varphi}_n(\omega)|^2 \leq C|\hat{\varphi}(\omega)|^2. \tag{5.22}$$

We already showed that $|\hat{\varphi}(\omega)|^2$ is integrable and hence a possible upper bound function. Since supp $\hat{\varphi}_n \subseteq [-2^n\pi, 2^n\pi]$ we may assume that $|\omega| \leq 2^n\pi$. From $\hat{\varphi}(\omega) = \frac{1}{\sqrt{2}}\hat{h}(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2})$ it follows that

$$|\hat{\varphi}(\omega)|^2 = |\hat{\varphi}_n(\omega)|^2 \left|\hat{\varphi}\left(\frac{\omega}{2^n}\right)\right|^2.$$

To prove (5.22) it is therefore sufficient to show that $|\hat{\varphi}(\omega)|^2 \geq \frac{1}{C}$ for $\omega \in [-\pi, \pi]$.

Let us first investigate the ε-neighbourhood of $\omega = 0$. Since $\hat{h} \in C[-\varepsilon, \varepsilon]$ and $|\hat{h}(\omega)|^2 \leq 2 = |\hat{h}(0)|^2$, the derivatives of the functions $|\hat{h}(\omega)|^2$ and $\ln|\hat{h}(\omega)|^2$ exist and vanish at $\omega = 0$. So there exists a $\delta > 0$ such that for all $|\omega| \leq \delta$:

$$0 \geq \ln\left(\frac{|\hat{h}(\omega)|^2}{2}\right) \geq -|\omega|.$$

Hence, for $|\omega| \leq \delta$

$$\begin{aligned} |\hat{\varphi}(\omega)|^2 &= \exp\left[\sum_{p=1}^{\infty} \ln\left(\frac{|\hat{h}(2^{-p}\omega)|^2}{2}\right)\right] \\ &\geq \exp\left[-\sum_{p=1}^{\infty} |2^{-p}\omega|\right] = e^{-|\omega|} \geq e^{-\delta}. \end{aligned} \tag{5.23}$$

Next, we analyze the behaviour for ω with $\pi \geq |\omega| > \delta$. Let $l \in \mathbb{N}$ such that $2^{-l} < \delta$. From (5.15) we know that $K := \inf_{\omega\in[-\frac{\pi}{2},\frac{\pi}{2}]} |\hat{h}(\omega)| > 0$, so we can estimate that

$$|\hat{\varphi}(\omega)|^2 = \prod_{p=1}^{l} \frac{|\hat{h}(\omega)|^2}{2} \left|\hat{\varphi}\left(\frac{\omega}{2^l}\right)\right|^2 \geq \frac{K^{2l}}{2^l} e^{-\delta} =: \frac{1}{C},$$

which finally proves the upper bound (5.22) for $|\hat{\varphi}(\cdot)|^2$.

Now, we have that $\{\varphi(t-k)\}_{k\in\mathbb{Z}}$ is orthonormal. A simple change of variable shows that $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ is othonormal for all $j \in \mathbb{Z}$.

Proof that $\{V_j\}_{j\in\mathbb{Z}}$ forms a multiresolution We have to show the 6 properties of an MRA in Definition 5.5. The multiresolution properties 1 and 3 are clearly true. The definition of V_0 yields Property 6. We are still lacking Properties 2, 4 and 5. Let us start with the causality $V_j \subset V_{j+1}$. From the refinement relation

$$\varphi(t) = \sum_{k=-\infty}^{\infty} \sqrt{2}h[k]\varphi(2t-k)$$

we conclude that

$$\varphi_{j,n}(t) = 2^{j/2}\varphi(2^j t - n) = \sum_{k=-\infty}^{\infty} h[k]2^{j/2}\varphi(2^{j+1}t - 2n - k) = \sum_{p=-\infty}^{\infty} h[p-2n]\varphi_{j+1,p},$$

where we substituted $p = 2n + k$. Consequently $V_j \subset V_{j+1}$.

To prove Property 4 we must show that any $u \in L^2(\mathbb{R})$ satisfies $\lim_{j\to-\infty} \|P_j u\| = 0$. Since $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_j we get

$$\|P_j u\| = \sum_{k=-\infty}^{\infty} |\langle u, \varphi_{j,k}\rangle|^2.$$

Suppose first that $|u(t)| \le A$ for all $t \in \mathbb{R}$ and supp $u \subseteq [2^{-J}, 2^J]$. Then

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |\langle u, \varphi_{j,k}\rangle|^2 &\le 2^j \left[\sum_{k=-\infty}^{\infty} \int_{-2^J}^{2^J} |u(t)||\varphi(2^j t - k)| \,\mathrm{d}t\right]^2 \\
&\le 2^j A^2 \left[\sum_{k=-\infty}^{\infty} \int_{-2^J}^{2^J} |\varphi(2^j t - k)| \,\mathrm{d}t\right]^2 \\
&\le 2^{J+1} A^2 \sum_{k=-\infty}^{\infty} \int_{-2^J}^{2^J} |\varphi(2^j t - k)|^2 2^j \,\mathrm{d}t \\
&= 2^{J+1} A^2 \int_{S_j} |\varphi(t)|^2 \,\mathrm{d}t,
\end{aligned}$$

with $S_j = \bigcup_{k\in\mathbb{Z}} [k - 2^{J+j}, k + 2^{J+j}]$ for $j < -J$. If $j \to -\infty$ then the measure of S_j tends to zero and by the dominated convergence theorem we have

$$\lim_{j\to-\infty} \sum_{k=-\infty}^{\infty} |\langle u, \varphi_{j,k}\rangle|^2 = 0.$$

Since the set of bounded functions with compact support is dense in $L^2(\mathbb{R})$ we get $\lim_{j\to-\infty} \|P_j u\| = 0$ for all $u \in L^2(\mathbb{R})$ and finally $\lim_{j\to-\infty} V_j = \{0\}$.

For the multiresolution property 5 we have to show that for any $u \in L^2(\mathbb{R})$

$$\lim_{j\to\infty} \|u - P_j u\|^2 = \lim_{j\to\infty} (\|f\|^2 - \|P_j f\|^2) = 0. \tag{5.24}$$

Let us only consider those u with supp $\hat{u} \subseteq [-2^J\pi, 2^J\pi]$ for sufficiently large J. We have already computed in (5.18) that the Fourier transform of $P_j u$ is

$$\widehat{P_j u}(\omega) = \hat{\varphi}(2^{-j}\omega) \sum_{k=-\infty}^{\infty} \hat{u}(\omega - 2^{j+1}k\pi) \cdot \hat{\varphi}^*(2^{-j}[\omega - 2^{j+1}k\pi]).$$

If $j > J$ then the supports of $\hat{u}(\cdot - 2^{j+1}k\pi)$ are disjoint for different $k \in \mathbb{Z}$ and

$$\begin{aligned}\|P_j u\|^2 &= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} |\hat{u}(\omega)|^2 |\hat{\varphi}(2^{-j}\omega)|^4 \,\mathrm{d}\omega \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |\hat{u}(\omega - 2^{j+1}k\pi)|^2 |\hat{\varphi}(2^{-j}\omega)|^2 |\hat{\varphi}(2^{-j}[\omega - 2^{j+1}k\pi])|^2 \,\mathrm{d}\omega\right]. \\ &\geq \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(\omega)|^2 |\hat{\varphi}(2^{-j}\omega)|^4 \,\mathrm{d}\omega\end{aligned}$$

We know that $|\hat{\varphi}(\omega)| \leq 1$ and (5.23) supplies for sufficiently small ω the inequality $|\hat{\varphi}(\omega)| \geq e^{-|\omega|/2}$, hence

$$\lim_{\omega\to 0} |\hat{\varphi}(\omega)| = 1.$$

Since $|\hat{u}(\omega)|^2|\hat{\varphi}(2^{-j}\omega)|^4 \leq |\hat{u}(\omega)|^2$ and $\lim_{j\to\infty} |\hat{u}(\omega)|^2|\hat{\varphi}(2^{-j}\omega)|^4 = |\hat{u}(\omega)|^2$ one can apply the dominated convergence theorem, which yields

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} |\hat{u}(\omega)|^2|\hat{\varphi}(2^{-j}\omega)|^4 \,\mathrm{d}\omega = \int_{-\infty}^{\infty} |\hat{u}(\omega)|^2 \,\mathrm{d}\omega = \|\hat{u}\|^2 = 2\pi\|u\|^2.$$

Therefore $\|P_j u\|^2 \geq \|u\|^2$ and on the other hand the orthogonality of the projector P_j implies $\|P_j u\| \leq \|u\|$ which finally proves

$$\lim_{j\to\infty} \|P_j u\|^2 = \|u\|.$$

Again, the result may be extended to the complete $L^2(\mathbb{R})$ by a density argument. □

5.4 Construction of Orthogonal Wavelet Bases

Similar to Section 5.2 we want to construct orthogonal wavelet bases. Since V_j is included in V_{j+1} we can form the orthogonal complement of V_j in V_{j+1}

$$V_{j+1} = V_j \oplus W_j, \quad \text{with} \quad W_j \perp V_j.$$

Now, we are looking for a function $\psi \in L^2(\mathbb{R})$ such that

$$\overline{\mathrm{span}\{\Psi_{j,k}\}} = W_j \quad \text{and} \quad \langle \Psi_{j,k}, \Psi_{j,k'}\rangle = \delta_{k,k'},$$

where $\Psi_{j,k} = 2^{j/2}\psi(2^j t - k)$ for $j, k \in \mathbb{Z}$. Since $\Psi \in V_1$ our desired representation is

$$\Psi(t) = \sum_{k=-\infty}^{\infty} g[k]\sqrt{2}\varphi(2t-k). \tag{5.25}$$

Let us define

$$\hat{g}(\omega) = \sum_{k=-\infty}^{\infty} g[k]e^{-ik\omega},$$

then the Fourier transform of (5.25) yields

$$\hat{\Psi}(2\omega) = \frac{1}{\sqrt{2}}\hat{g}(\omega)\hat{\varphi}(\omega). \tag{5.26}$$

Lemma 5.14 *The family $\{\Psi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j if and only if*

$$|\hat{g}(\omega)|^2 + |\hat{g}(\omega+\pi)|^2 = 2 \tag{5.27}$$

and

$$\hat{g}(\omega)\hat{h}^*(\omega) + \hat{g}(\omega+\pi)\hat{h}^*(\omega+\pi) = 0. \tag{5.28}$$

Proof: The lemma will be proven for $j = 0$ from which it is extensible to the case $j \neq 0$ via an appropriate scaling. As for (5.9) one can verify that $\{\Psi(t-k)\}_{k\in\mathbb{Z}}$ is orthonormal if and only if

$$I(\omega) = \sum_{k=-\infty}^{\infty} |\hat{\Psi}(\omega + 2k\pi)|^2 = 1, \quad \forall\omega \in \mathbb{R}. \tag{5.29}$$

Since $\hat{\Psi}(\omega) = 2^{-1/2}\hat{g}(\omega/2)\hat{\varphi}(\omega/2)$ and $\hat{g}(\omega)$ is 2π periodic,

$$\begin{aligned} I(\omega) &= \frac{1}{2}\sum_{k=-\infty}^{\infty} \left|\hat{g}\left(\frac{\omega}{2}+k\pi\right)\right|^2 \left|\hat{\varphi}\left(\frac{\omega}{2}+k\pi\right)\right|^2 \\ &= \frac{1}{2}\left[\left|\hat{g}\left(\frac{\omega}{2}\right)\right|^2 \sum_{p=-\infty}^{\infty}\left|\hat{\varphi}\left(\frac{\omega}{2}+2p\pi\right)\right|^2 + \left|\hat{g}\left(\frac{\omega}{2}+\pi\right)\right|^2 \sum_{p=-\infty}^{\infty}\left|\hat{\varphi}\left(\frac{\omega}{2}+(2p+1)\pi\right)\right|^2\right]. \end{aligned}$$

Now, from

$$\sum_{p=-\infty}^{\infty} |\hat{\varphi}(\omega+2p\pi)|^2 = 1$$

we have shown that (5.27) and (5.29) are equivalent.

Next we investigate $V_0 \perp W_0$ which is equivalent to the condition

$$\langle\Psi(\cdot), \varphi(\cdot - k)\rangle = (\Psi * \varphi^*)(k) = 0.$$

The sampled sequence $\Psi * \varphi^*$ is zero if its Fourier series (see Proposition 2.2) satisfies

$$\sum_{k=-\infty}^{\infty} \hat{\Psi}(\omega+2k\pi)\hat{\varphi}^*(\omega+2k\pi) = 0, \quad \forall\omega\in\mathbb{R}.$$

With the same argumentation as above we prove that the latter equation is equivalent to (5.28).

Finally we show that $V_0 \oplus W_0 = V_1$. Since $\{\sqrt{2}\varphi(2\cdot -k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_1 and $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$, $\{\Psi(\cdot - k)\}_{k\in\mathbb{Z}}$ are orthonormal bases of V_0 and W_0, respectively, we have to show that for any $a[\cdot] \in l^2(\mathbb{Z})$ there exist $b[\cdot], c[\cdot] \in l^2(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty} a[k]\sqrt{2}\varphi(2t-k) = \sum_{k=-\infty}^{\infty} b[k]\varphi(t-k) + \sum_{k=-\infty}^{\infty} c[k]\Psi(t-k). \tag{5.30}$$

Taking the Fourier transform of (5.30) yields

$$\frac{1}{\sqrt{2}}\hat{a}\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right) = \hat{b}(\omega)\hat{\varphi}(\omega) + \hat{c}(\omega)\hat{\Psi}(\omega).$$

Inserting $\hat{\Psi}(\omega) = 2^{-1/2}\hat{g}(\omega/2)\hat{\varphi}(\omega/2)$ and $\hat{\varphi}(\omega) = 2^{-1/2}\hat{h}(\omega/2)\hat{\varphi}(\omega/2)$ shows that (5.30) is necessarily satisfied if

$$\hat{a}\left(\frac{\omega}{2}\right) = \hat{b}(\omega)\hat{h}\left(\frac{\omega}{2}\right) + \hat{c}(\omega)\hat{g}\left(\frac{\omega}{2}\right). \tag{5.31}$$

Let us set $b[\cdot]$ and $c[\cdot]$ so that

$$\begin{aligned}\hat{b}(2\omega) &:= \frac{1}{2}[\hat{a}(\omega)\widehat{\overline{h}}(\omega) + \hat{a}(\omega+\pi)\widehat{\overline{h}}(\omega+\pi)],\\ \hat{c}(2\omega) &:= \frac{1}{2}[\hat{a}(\omega)\widehat{\overline{g}}(\omega) + \hat{a}(\omega+\pi)\widehat{\overline{g}}(\omega+\pi)].\end{aligned}$$

Calculating the right-hand side of (5.31) one easily verifies the equality to the left-hand side by inserting (5.27), (5.28) and using

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega+\pi)|^2 = 2. \tag{5.32}$$

Since $\hat{b}(\omega)$ and $\hat{c}(\omega)$ are 2π-periodic functions they are the Fourier series of two sequences $b[\cdot]$ and $c[\cdot]$ that satisfy (5.30) which completes the proof. □

The following theorem shows how to construct an orthonormal basis of W_j by scaling and translating a wavelet Ψ.

Theorem 5.15 Mallat, Meyer
Let $\varphi \in L^2(\mathbb{R})$ be an orthogonal scaling function and h the corresponding conjugate mirror filter. Then Ψ defined by

$$\hat{\Psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right), \tag{5.33}$$

where

$$\hat{g}(\omega) = e^{-i\omega}\hat{h}^*(\omega+\pi), \tag{5.34}$$

is an orthogonal wavelet, i.e.,

$$\{\Psi_{j,k}(t) = 2^{j/2}\Psi(2^j t - k)\}_{k\in\mathbb{Z}}$$

form orthonormal bases of W_j for all $j \in \mathbb{Z}$ and $\{\Psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Proof: Since

$$\Psi(t) = \sum_{k=-\infty}^{\infty} g[k]\sqrt{2}\varphi(2t-k) \in V_1,$$

and on the other hand

$$\Psi(t) = \sum_{k=-\infty}^{\infty} \langle \Psi, \varphi_{1,k}\rangle \varphi_{1,k},$$

we conclude that

$$g[k] = \langle \Psi, \varphi_{1,k}\rangle = \sqrt{2}\langle \Psi, \varphi(2\cdot -k)\rangle.$$

In order to show that $\{\Psi(\cdot - k) : k \in \mathbb{Z}\}$ forms an orthonormal bases of W_0 we involve the previous lemma. Indeed, the function $\hat{g}(\omega) = e^{-i\omega}\hat{h}^*(\pi + \omega)$ satisfies

$$|\hat{g}(\omega)|^2 + |\hat{g}(\omega + \pi)|^2 = |\hat{h}(\omega + \pi)|^2 + |\hat{h}(\omega)|^2 = 2,$$

and

$$\hat{g}(\omega)\hat{h}^*(\omega) + \hat{g}(\omega + \pi)\hat{h}^*(\omega + \pi) = (e^{-i\omega} + e^{-i(\omega+\pi)})\hat{h}^*(\omega)\hat{h}^*(\omega + \pi) = 0.$$

Finally we show that $\{\Psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. From $W_0 \perp V_0$ we conclude that

$$\langle \Psi_{j,k}, \varphi_{j,k'} \rangle = \langle \Psi(\cdot - k), \varphi(\cdot - k') \rangle = 0,$$

i.e. $W_j \perp V_j$. Therefore, $W_j \perp W_l$ for all $j \neq l$. Since $\{V_j : j \in Z\}$ forms a multiresolution analysis the proof is completed by observing that $\text{span}\{\Psi_{j,k} : j, k \in \mathbb{Z}\}$ is dense in $L^2(\mathbb{R})$. □

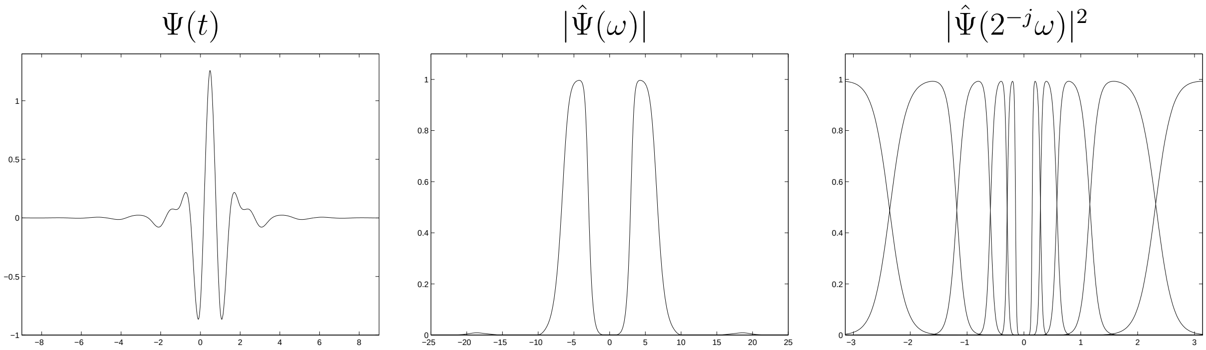


Figure 5.3: Cubic spline wavelet, its Fourier transform and the graphs of $|\hat{\Psi}(2^{-j}\omega)|^2$ for $1 \leq j \leq 5$.

The proof points out that

$$g[k] = \langle \Psi, \varphi_{1,k} \rangle = \sqrt{2}\langle \Psi(\cdot), \varphi(2 \cdot -k) \rangle,$$

when

$$\Psi(t) = \sum_{k=-\infty}^{\infty} g[k]\sqrt{2}\varphi(2t - k).$$

Since $\hat{g}(\omega) = e^{-i\omega}\overline{\hat{h}(\omega + \pi)}$ is the Fourier transform of g we get the following useful equality:

$$\boxed{g[k] = (-1)^{1-k}h[1 - k]}$$

Remark 5.16 It can be shown that for all $\omega \in \mathbb{R}\backslash\{0\}$

$$\sum_{j=-\infty}^{\infty} |\Psi(2^j\omega)|^2 = 1.$$

This is illustrated in Figure 5.3. According to the previous theorem any $f \in L^2(\mathbb{R})$ can be represented as

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \Psi_{j,k} \rangle \Psi_{j,k} = \sum_{j=-\infty}^{\infty} Q_j f,$$

where $Q_j = (P_{j+1} - P_j)f$. The calculations to obtain the coefficients of a signal decomposed in a certain wavelet basis may can be done by fast wavelet transform algorithms.

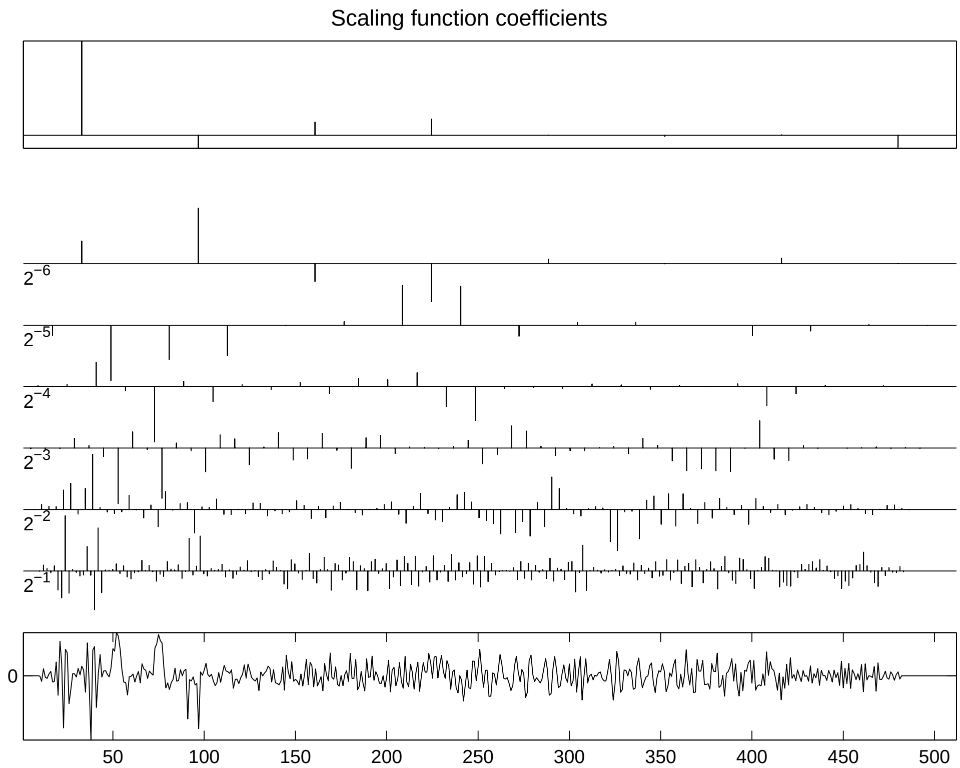


Figure 5.4: Wavelet coefficients $d_j[n] = \langle f, \Psi_{j,n} \rangle$ calculated at scales $2^j, j = -1, \ldots, -6$ with the cubic spline wavelet. At the top is the remaining coarse signal approximation $a_J[n] = \langle f, \varphi_{-5,n} \rangle$.

Example 5.17 Examples of orthogonal basis functions

1. Box function

$$h[k] = \begin{cases} \frac{1}{\sqrt{2}}, & k = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

2. B-splines of order d generate the so called Battle-Lemarié wavelets

$$\hat{h}(\omega) = e^{i\epsilon\omega/2} \sqrt{\frac{S_{2d}(\omega)}{2^{2d-1} S_{2d}(2\omega)}},$$

where $\epsilon = 0$ for even d and $\epsilon = 1$ otherwise. For the definition of S_{2d} see example 5.12. For the special case of piecewise linear orthogonal scaling functions we obtain

$$\begin{aligned}\hat{h}(\omega) &= \sqrt{2}\left[\frac{1+2\cos^2\frac{\omega}{2}}{1+2\cos\omega}\right]^{1/2}\cos^2\frac{\omega}{2},\\ \hat{g}(\omega) &= e^{-i\omega}\overline{\hat{h}(\omega+\pi)}.\end{aligned}$$

Remark 5.18 Counter example

Let

$$\hat{h}(\omega) = \sqrt{2}\cos\left(\frac{3}{2}\omega\right),$$

then

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega+\pi)|^2 = 2\cos^2\left(\frac{3}{2}\omega\right) + 2\cos\left(\frac{3}{2}\omega + \frac{3}{2}\pi\right) = 2.$$

But it can be shown that

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty}\frac{\hat{h}(2^{-j}\omega)}{\sqrt{2}} = \frac{1}{3}\chi_{[-\frac{3}{2},\frac{3}{2}]}(\omega),$$

which is obviously not an orthogonal scaling function.

5.5 Daubechies Compactly supported wavelets

The scaling function φ has compact support if and only if the associated conjugate mirror filter $h[\cdot]$ has compact support, i.e., there exists an $N \in \mathbb{N}$ such that

$$\hat{h}(\omega) = \sum_{k=0}^{N-1} h[k]e^{-ik\omega}.$$

Compactly supported filters $h[\cdot]$ have been computed by Ingrid Daubechies. For further details see [2] or attend a more advanced course on wavelets.

5.6 Fast Wavelet Transform

Decomposition coefficients in a wavelet orthogonal basis are computed with a fast algorithm that cascades discrete convolutions with h and g, and subsamples the output. A fast Wavelet transform decomposes successively each approximation $P_{j+1}f$ into a coarser approximation $P_j f$ plus the wavelet coefficients carried by $P_{j+1}f$.

Because of $V_{j+1} = V_j \oplus W_j$ a function $f \in V_{j+1}$ may be represented either by an expansion with respect to a scaling function basis

$$f = \sum_{k=-\infty}^{\infty}\langle f, \varphi_{j+1,k}\rangle\varphi_{j+1,k} = \sum_{k=-\infty}^{\infty} a_{j+1}[k]\varphi_{j+1,k},$$

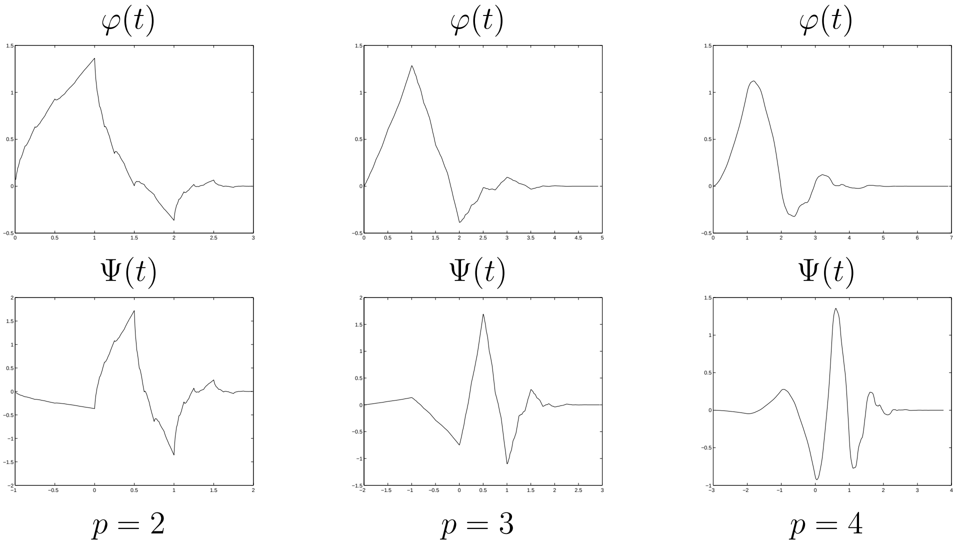


Figure 5.5: Daubechies scaling function φ and wavelets Ψ with p vanishing moments.

or with respect to orthogonal bases of V_j and W_j

$$\begin{aligned} f &= \sum_{k=-\infty}^{\infty} \langle f, \varphi_{j,k} \rangle \varphi_{j,k} + \sum_{k=-\infty}^{\infty} \langle f, \Psi_{j,k} \rangle \Psi_{j,k} \\ &= \sum_{k=-\infty}^{\infty} a_j[k] \varphi_{j,k} + \sum_{k=-\infty}^{\infty} d_j[k] \Psi_{j,k}. \end{aligned}$$

Theorem 5.19 Mallat

At the decomposition

$$a_j[n] = \sum_{k=-\infty}^{\infty} h[k-2n] a_{j+1}[k] = (a_{j+1} * h^*)[2n], \tag{5.35}$$

$$d_j[n] = \sum_{k=-\infty}^{\infty} g[k-2n] a_{j+1}[k] = (a_{j+1} * g^*)[2n], \tag{5.36}$$

and at the reconstruction

$$\begin{aligned} a_{j+1}[n] &= \sum_{k=-\infty}^{\infty} h[n-2k] a_j[k] + \sum_{k=-\infty}^{\infty} g[n-2k] d_j[k] \qquad (5.37) \\ &= (\check{a}_j * h)[n] + (\check{d}_j * g)[n], \end{aligned}$$

where $\check{x}[k] = \begin{cases} x[k], & \textit{if } k = 2p,\ p \in \mathbb{Z}, \\ 0, & \textit{otherwise}, \end{cases}$ *denotes the oversampling.*

Proof:

Proof of (5.35) Resulting from

$$a_j[n] = \langle f, \varphi_{j,n} \rangle = \sum_{k=-\infty}^{\infty} \langle f, \varphi_{j+1,k} \rangle \langle \varphi_{j,n}, \varphi_{j+1,k} \rangle,$$

and with $x = 2^j t - n$,

$$\begin{aligned} \langle \varphi_{j,n}, \varphi_{j+1,k} \rangle &= \int_{-\infty}^{\infty} 2^{\frac{j}{2}} \varphi(2^j t - n) 2^{\frac{j+1}{2}} \varphi^*(2^{j+1} t - k) \, \mathrm{d}t \\ &= \sqrt{2} \int_{-\infty}^{\infty} \varphi(x) \varphi^*(2x - k + 2n) \, \mathrm{d}x \\ &= h[k - 2n], \end{aligned}$$

we conclude

$$\varphi_{j,n} = \sum_{k=-\infty}^{\infty} h[k-2n] \varphi_{j+1,k}.$$

Taking the inner product with f on both sides of this equality yields (5.35).

Proof of (5.36) Analogously $\langle \Psi_{j,n}, \varphi_{j+1,n} \rangle = g[k - 2n]$ proves

$$\Psi_{i,n} = \sum_{k=-\infty}^{\infty} g[k-2n] \varphi_{j+1,k}$$

which results in (5.36).

Proof of (5.37) With the formulas from above we obtain

$$\begin{aligned} \varphi_{j+1,n} &= \sum_{k=-\infty}^{\infty} \langle \varphi_{j+1,n}, \varphi_{j,k} \rangle \varphi_{j,k} + \sum_{k=-\infty}^{\infty} \langle \varphi_{j+1,n}, \Psi_{j,k} \rangle \Psi_{j,k} \\ &= \sum_{k=-\infty}^{\infty} h[k-2n] \varphi_{j,k} + \sum_{k=-\infty}^{\infty} g[k-2n] \Psi_{j,k}. \end{aligned}$$

Again, taking the inner product with f on both sides completes the proof. □

The decomposition and reconstruction,

$$a_{j+1}[\cdot] \begin{array}{l} \nearrow \; a_j[\cdot] \\ \searrow \; d_j[\cdot] \end{array} \quad \text{and} \quad \begin{array}{r} a_j[\cdot] \; \searrow \\ d_j[\cdot] \; \nearrow \end{array} a_{j+1}[\cdot]$$

are nothing but changes of basis functions.

Iterating the decomposition yields for given coefficients $a_{j+1}[\cdot]$,

$$\begin{array}{ccccccccc} \boxed{a_{j+1}[\cdot]} & \longrightarrow & a_j[\cdot] & \longrightarrow & a_{j-1}[\cdot] & \longrightarrow & \ldots \; a_1[\cdot] & \longrightarrow & \boxed{a_0[\cdot]} \\ & \searrow & & \searrow & & \searrow & \ldots & \searrow & \\ & & \boxed{d_j[\cdot]} & & \boxed{d_{j-1}[\cdot]} & & \ldots & & \boxed{d_0[\cdot]} \end{array}$$

the coefficients

$$(D[l,k])_{l=-1,\ldots,j} := (a_0[\cdot], d_0[\cdot], d_1[\cdot], \ldots, d_j[\cdot]).$$

The translation $(a_{j+1}[k]) \to (D[l,k])$ is called the **discrete Wavelet transform**. The backward transform is provided by the reconstruction

$$\begin{array}{ccccccccc}
\boxed{d_0[\cdot]} & & \boxed{d_1[\cdot]} & & \ldots & \boxed{d_{j-1}[\cdot]} & & \boxed{d_j[\cdot]} & & \\
& \searrow & & \searrow & \ldots & & \searrow & & \searrow & \\
\boxed{a_0[\cdot]} & \longrightarrow & a_1[\cdot] & \longrightarrow & \ldots & a_{j-1}[\cdot] & \longrightarrow & a_j[\cdot] & \longrightarrow & \boxed{a_{j+1}[\cdot]}
\end{array}$$

In practice the signal a_{j+1} is 2^{j+1} periodic which means that we have $N = 2^{j+1}$ coefficients. Then the discrete Wavelet transform requires $O(N)$ algebraic operations.

Index