## Exercise Sheet 7.

Hand in due to: **Friday, 5/05/2023, 14:00**

**Exercise 1** (Translational and radial invariance | 4 points).

Show that

(a) A kernel $K$ is translational invariant if and only if there exists $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $K(x, y) = \Phi(x - y)$ for all $x, y \in \mathbb{R}^d$.

(b) A kernel $K$ is radial if and only if there exists $\Phi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ such that $K(x, y) = \Phi(\|x - y\|_2)$ for all $x, y \in \mathbb{R}^d$.

(c) Let $K(x, y) = \Phi(x - y)$ be a positive definite and translational invariant kernel on $\mathbb{R}^d$. Then $\widehat{\Phi}(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d$.

(d) Let $K(x, y) = \Phi(x - y)$ be a positive definite and translational invariant kernel on $\mathbb{R}^d$. Then $\Phi(0) \geq 0$ and $\Phi(0) = 0$ implies $\Phi(x) = 0$ for $x \in \mathbb{R}^d$.

**Exercise 2** (Approximation by convolution | 4 points).

Let $m \in \mathbb{N}$ and $g_m(x) := (m/\pi)^{d/2} \exp(-m\|x\|_2^2)$, $x \in \mathbb{R}^d$. Show that:

(a) $\displaystyle\int_{\mathbb{R}^d} g(x)dx = 1$

(b) $\displaystyle\widehat{g}_m(x) = \frac{1}{(2\pi)^{d/2}} \exp\Big(-\frac{\|x\|_2^2}{4m}\Big)$

(c) $\widehat{\widehat{g}}_m(x) = g_m(x)$

(d) If $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is continuous and slowly increasing, i.e., there exist $l \in \mathbb{N}$ and $M > 0$ such that $|f(x)| \leq M(1 + \|x\|_2)^l$ for all $x \in \mathbb{R}^d$, then

$$f(x) = \lim_{m\to\infty} (f * g_m)(x) := \lim_{m\to\infty} \int_{\mathbb{R}^d} f(y)g_m(y - x)dy.$$

Hint. *Consider first the case $x = 0$, then conclude the statement for $x \neq 0$ by replacing $f$ by $f(\cdot + x)$.*

**Exercise 3** (Trace norm | 4 points).

Let $A = [a_{ij}]_{i,j=1}^n \in \mathbb{R}^{n\times n}$ be a symmetric and positive semi-definite matrix. Show that the trace norm $\|A\|_{\mathrm{tr}} := \sum_{i=1}^n a_{ii}$ satisfies the following three norm properties on the set of positive semi-definite matrices:

(a) $\|A\|_{\mathrm{tr}} = 0$ if and only if $A = 0$

(b) $\|\alpha A\|_{\mathrm{tr}} = |\alpha|\|A\|_{\mathrm{tr}}$

(c) $\|A + B\|_{\mathrm{tr}} \leq \|A\|_{\mathrm{tr}} + \|B\|_{\mathrm{tr}}$

Hint. *First prove that $|a_{ij}| \leq \sqrt{a_{ii}a_{jj}}$.*

**Exercise 4** (Kahan matrix | 4 points).

The Kahan matrix is given by

$$K_n(\theta) = [k_{ij}]_{i,j=1}^n = \operatorname{diag}(1, s, s^2, \dots, s^{n-1}) \begin{bmatrix} 1 & -c & \cdots & -c \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -c \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

where $c = \cos(\theta)$, $s = \sin(\theta)$ and $0 < \theta < \pi/2$. Show that:

(a) The inverse of Kahan matrix is given by the upper triangular matrix $K_n^{-1} = [r_{ij}]_{i,j=1}^n$ with

$$r_{ij} = \begin{cases} s^{1-j}, & \text{if } i = j, \\ s^{1-j}c(c+1)^{j-i-1}, & \text{if } i < j. \end{cases}$$

(b) There holds

$$s^{n-1}K_n^{-1}(\theta) \longrightarrow [0, \dots, 0, \mathbf{x}] \text{ as } \theta \longrightarrow 0,$$

where $\mathbf{x} = [2^{n-2}, 2^{n-3}, \dots, 1]^\top$.

(c) The spectral norm of the inverse Kahan matrix satisfies

$$|k_{nn}| \, \|K_n^{-1}\|_2 \longrightarrow \sqrt{\frac{4^{n-1}+2}{3}} \text{ as } \theta \longrightarrow 0.$$