# Exercise Sheet 6.

Hand in due to: **Friday, 28/04/2023, 14:00**

**Exercise 1** (Sobolev space as a RKHS | 4 points).

Show that $H_0^1\big((0,1)\big)$ is a RKHS with the covariance

$$K(x,y) := \min(x,y) - xy.$$

of the Brownian bridge as reproducing kernel.

**Exercise 2** (Basic operations in $H_K(\Omega)$ | 4 points).

Let $K$, $K_1$ and $K_2$ be symmetric and positive definite kernels on $\Omega \times \Omega \longrightarrow \mathbb{R}$. Consider a function $g : \Omega \longrightarrow \mathbb{R} \setminus \{0\}$ and a constant $a > 0$. Show that:

(a) $K(x,y) := g(x)g(y)$ results in $H_K(\Omega) = \operatorname{span}\{g\}$.

(b) $K(x,y) := aK_1(x,y)$ results in $H_K(\Omega) = H_{K_1}(\Omega)$.

(c) $K(x,y) := g(x)K_1(x,y)g(y)$ results in $H_K(\Omega) = gH_{K_1}(\Omega) := \{gf \mid f \in H_{K_1}(\Omega)\}$.

(d) $K(x,y) := K_1(x,y) + K_2(x,y)$ results in $H_K(\Omega) = H_{K_1}(\Omega) + H_{K_2}(\Omega)$.

**Exercise 3** (Interpolation in $H_K(\Omega)$ | 4 points).

Let $H_K(\Omega)$ be an RKHS on $\Omega$ with reproducing kernel $K$ and $f \in H_K(\Omega)$. For a set $X$ of pairwise distinct points, let $P_{H_X}$ denote the corresponding power function. Show that:

(a) $\|f\|^2_{H_K(\Omega)} = \|f_X\|^2_{H_K(\Omega)} + \|f - f_X\|^2_{H_K(\Omega)}$.

(b) If $Y \subset \Omega$ is finite and $X \subset Y$, then $P_{H_Y}(x) \leq P_{H_X}(x)$ for all $x \in \Omega \setminus X$.

**Exercise 4** (Power function for derivatives | 4 points).

Let $H_K(\Omega)$ be an RKHS on $\Omega$ with reproducing kernel $K \in C^{2k}(\Omega \times \Omega)$ and $f \in H_K(\Omega)$. For a set $X$ of pairwise distinct points, we define the generalized power function as

$$P_{H_X}^{(\alpha)}(x) := \|D_2^\alpha K(\cdot,x) - D_2^\alpha K_X(\cdot,x)\|_{H_K(\Omega)}.$$

Show that for all $x \in \Omega$

$$|D_2^\alpha f(x) - D_2^\alpha f_X(x)| \leq P_{H_X}^{(\alpha)}(x)\|f\|_{H_K(\Omega)}.$$