## Exercise Sheet 5.

Hand in due to: **Friday, 21/04/2023, 14:00**

**Exercise 1** (Strictly positive definiteness $K$ | 4 points).

Let $H_K(\Omega)$ be a RKHS on $\Omega$ with reproducing kernel $K$. Prove that $K$ is strictly positive definite if and only if $\{\delta_x : x \in \Omega\}$ are linearly independent.

**Exercise 2** (RKHS and smoothness | 4 points).

Let $\Omega$ be a nonempty set, $K : \Omega \times \Omega \longrightarrow \mathbb{R}$ be a strictly positive definite kernel, and $H_K$ corresponding RKHS. Show the following statements:

a) $d_K(x, y) = \|K(\cdot, x) - K(\cdot, y)\|_{H_K(\Omega)}$ is a metric.

b) If $\dim(\Omega) = \infty$, then $\dim(H_K(\Omega)) = \infty$.

c) Every $f \in H_K(\Omega)$ is Lipschitz continuous with respect to $d_K$.

d) If $\Omega \subset \mathbb{R}^d$ is open and $K \in C^{2k}(\Omega \times \Omega)$ for $k \in \mathbb{N}$, then $H_K(\Omega) \subset C^k(\Omega)$.

**Exercise 3** (Hierarchical Matrix | 4 points).

Let $N = 2^J$ and let $\mathbf{T}$ be an $(N \times N)$-hierarchical matrix with a maximal Level $L$ as on programming sheet 2. Assume that the low rank approximations of the offdiagonal blocks are of rank $\leq m$.

a) Show that for storing $\mathbf{T}$, one must store $mL2^J + 2^{2J-L}$ numbers.

b) Show that the matrix-vector multiplication $\mathbf{Tx}$ requires $2mL2^J + 2^{2J-L}$ multiplications.

**Exercise 4** (Regularity of the Fourier transform | 4 points).

Let $f \in L^1(\mathbb{R})$. Prove that

a) $\hat{f} \in L^\infty(\mathbb{R})$,

b) $\hat{f}$ is uniformly continuous on $\mathbb{R}$.