Exercise Sheet 4.

Hand in due to: **Friday, 14/04/2023, 14:00**

Exercise 1 (Fourier transform properties | 4 points).

Let $\widehat{u}$ be a Fourier transform of $u : \mathbb{R}^d \longrightarrow \mathbb{R}$. Prove then following properties of $\widehat{u}$.

(a) $\widehat{\overline{u(x)}} = \overline{\widehat{u}(-\xi)}$.

(b) $\widehat{u(-x)} = \widehat{u}(-\xi)$.

(c) If u is even or odd, then $\widehat{u}$ is even or odd.

(d) If u is real, then $\widehat{u}$ has even real part and odd imaginary part.

(e) If u is real and even, then $\widehat{u}$ is real and even.

(f) If u is real and odd, then $\widehat{u}$ is imaginary and even.

(g) Let be $a \neq 0$ real, then $\widehat{u(ax)} = |a|^{-1}\widehat{u}(a^{-1}\xi)$.

(h) For $y \in \mathbb{R}^d$, one has $\widehat{u(x-y)} = e^{-i\langle x,\xi\rangle}\widehat{u}(\xi)$.

Exercise 2 (Gaussian kernel | 4 points).

Prove that $K(x, y) = \exp(-\|x - y\|^2/(2\sigma)^2)$ is a reproducing kernel.

Hint. *By using the positive definiteness, prove that the sum and the multiplication of two kernels define also kernels.*

Exercise 3 (Mercer's theorem | 4 points).

Let H be a Hilbert space of functions $u : \Omega \longrightarrow \mathbb{R}$ and $A : H \longrightarrow H$ be a compact, positive definite, and self-adjoint operator with eigenpairs (λ_j, ϕ_j) such that

$$\Lambda^2 = \sum_{j=1}^{\infty} \lambda_j < \infty, \quad |\phi_j(x)| \leq c, \quad (\phi_i, \phi_j) = \delta_{ij} \quad \text{for all} \quad i, j \in \mathbb{N} \text{ and } x \in \Omega.$$

Let $H_- \supset H$ be a Hilbert space with a scalar product $(u, v)_- = (A^{1/2}u, A^{1/2}v)_H$, $H_+ = \operatorname{img}(A^{1/2})$ be a Hilbert space with a scalar product $(u, v)_+ = (A^{-1/2}u, A^{-1/2}v)_H$. Show that

(a) $H_+ \subset H \subset H_-$ is a Gelfand tripple. Additionally consider the case when $H = L^2(\Omega)$.

(b) H_+ is a RKHS with reproducing kernel $K(x, y) = \sum_{j=1}^{\infty} \lambda_j \overline{\phi_j(x)}\phi_j(y)$.

Exercise 4 (Non-continuous kernel | 4 points).

Let $\{a_n\}_{n=0}^{\infty} = \{1/2^n\}_{n=0}^{\infty}$ and ϕ_n be a function which vanishes outside (a_{n+1}, a_{n-1}) and equals 1 at a_n. Let $K(x, y) = \sum_{n=1}^{\infty} \phi_n(x)\phi_n(y)$. Show that

(a) $K(x, y)$ is a reproducing kernel.

(b) $K(x, \cdot), K(\cdot, y) \in C([0, 1])$, but $K(x, y) \notin C([0, 1] \times [0, 1])$, i.e. K is continuous separately in x and y but not continuous in both variables.

Hint. *Use the Arzelà–Ascoli theorem to show that the unit ball of the RKHS is not compact in* $C([0, 1])$. *Then conclude that* $K(x, y) \notin C([0, 1] \times [0, 1])$.