Exercise Sheet 3.

Hand in due to: **Friday, 31/03/2023, 14:00**

Exercise 1 (Riesz representation theorem | 4 points).

Let H be a Hilbert space and H^* the dual space.

(a) Prove that for every $\phi \in H^*$ there exists a unique $u_\phi \in H$ such that for all $v \in H$

$$\phi(v) = \langle u_\phi, v\rangle_H \quad \text{and} \quad \|u_\phi\|_H = \|\phi\|_{H^*}.$$

(b) Using (a), conclude that the dual space of a Hilbert space is also a Hilbert space.

Exercise 2 (Dirac delta function | 4 points).

Let H be a Hilbert space of continuous functions $f : X \longrightarrow \mathbb{R}$. Let the Dirac delta function $\delta_x : H \longrightarrow \mathbb{R}$ be the map from $f \in H$ to $f(x) \in \mathbb{R}$.

(a) Assume H is a reproducing kernel Hilbert space, then what is the Riesz representer of the function δ_x?

(b) Prove that H is a reproducing kernel Hilbert space if and only if δ_x is bounded.

(c) Show if $\lim\limits_{n\to\infty} \|f_n - f\|_H = 0$ and H is a reproducing kernel Hilbert space, then $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$.

(d) Is $L^2(\mathbb{R})$ a reproducing kernel Hilbert space?

Justify your answer.

Exercise 3 (Adaptive Cross Approximation I | 4 points).

Let $X = x_1, \dots, x_m$ and $Y = y_1, \dots, y_n$ be two sets of pairwise distinct points in $\mathbb{R}^d$ and D_X, D_Y the convex hulls of X and Y, respectively. Let us consider a function $f : D_x \times D_Y \longrightarrow \mathbb{R}$ of the type

$$f(x, y) = \sum_{k=1}^{N_p} g_k(x) h_k(y) + R_p(x, y),$$

where $|R_p(x, y)| \le \epsilon_p$ and $\epsilon_p \longrightarrow 0$ as $p \longrightarrow \infty$. We construct sequences by the following rule

$$r_0(x, y) = f(x, y), \quad s_0(x, y) = 0$$

and for $k = 0, 1, \dots$

$$r_{k+1}(x, y) = r_k(x, y) - \gamma_{k+1} r_k(x, y_{j_{k+1}}) r_k(x_{i_{k+1}}, y),$$

$$s_{k+1}(x, y) = s_k(x, y) + \gamma_{k+1} r_k(x, y_{j_{k+1}}) r_k(x_{i_{k+1}}, y),$$

where $\gamma_{k+1} = r_k(x_{i_{k+1}}, y_{j_{k+1}})^{-1}$ and $x_{i_{k+1}}$ and $y_{j_{k+1}}$ are chosen in every step so that $r_k(x_{i_{k+1}}, y_{j_{k+1}}) \neq 0$.

Let us also denote

$$f(x,[y]_k) = \begin{bmatrix} f(x,y_{j_1}) \\ \vdots \\ f(x,y_{j_k}) \end{bmatrix}, \quad f([x]_k,y) = \begin{bmatrix} f(x_{i_1},y) \\ \vdots \\ f(x_{i_k},y) \end{bmatrix}, \quad M_k^l(x) = \begin{bmatrix} f(x_{i_1},y_{j_1}) & \cdots & f(x_{i_1},y_{j_k}) \\ \vdots & & \vdots \\ f(x,y_{j_1}) & \cdots & f(x,y_{j_k}) \\ \vdots & & \vdots \\ f(x_{i_k},y_{j_1}) & \cdots & f(x_{i_k},y_{j_k}) \end{bmatrix}$$

and $M_k = M_k^l(x_{i_l})$.

(a) Show that for $1 \leq l \leq k$ and all $x \in D_X$ we have $r_k(x, y_{j_l}) = 0$.

(b) Applying (a), prove that for $1 \leq l < k$

$$\det M_k^l(x) = r_{k-1}(x_{i_k}, y_{j_k}) \det M_{k-1}^l(x) - r_{k-1}(x, y_{j_k}) \det M_{k-1}^l(x_{i_k})$$

holds and

$$\det M_1^1 = r_0(x, y_{j_1}), \quad \det M_k^k = r_{k-1}(x, y_{j_k}) \det M_{k-1}, \quad k > 1.$$

Especially

$$\det M_k = r_0(x_{i_1}, y_{j_1}) \cdot \cdots \cdot r_{k-1}(x_{i_k}, y_{j_k}).$$

Exercise 4 (Adaptive Cross Approximation II | 4 points).

Our goal for this exercise is to analyze the interpolation error. This is done in two steps.

(a) Show that for the functions r_{n_p} there holds that

$$r_{n_p} = E_p(f_x)(y) - \sum_{l=1}^{n_p} \frac{\det M_{n_p}^l(x)}{\det M_{n_p}} E_p(f_{x_{i_l}})(y),$$

where $f_x(y)$ denotes the function $f(x,y)$ for a fixed point $x \in D_X$ and $E_p(f_x)(y) = f_x(y) - L_{p-1}(f_x)(y)$ with the interpolation polynomial $L_{p-1}(f_x)$ of degree $p-1$ with respect to the points $y_{n_1}, \dots, y_{n_p}$.

Hint. *Use fact that* $s_k(x,y) = f(x,[y]_k)^\top M_k^{-1} f([x]_k, y)$.

(b) Assume we choose x_{i_k} such that

$$|r_{k-1}(x_{i_k}, y_{j_k})| \geq |r_{k-1}(x, y_{j_k})| \quad \text{for all} \quad x \in D_X.$$

Then, for $1 \leq l < k$ show that

$$\sup_{x \in D_X} \frac{|\det M_k^l(x)|}{|\det M_k|} \leq 2^{k-l}$$

Hint. *Use 3 (b).*

By applying the results above conclude the estimate

$$|f(x,y) - s_{n_p}(x,y)| \leq 2^{n_p} \sup_{x \in D_X} |E_p(f_x)(y)|.$$