Exercise Sheet 2.

Hand in due to: **Friday, 24/03/2023, 14:00**

Exercise 1 (Sobolev embedding theorem | 4 points).

Use the Fourier transform to prove that there holds $u \in L^\infty(\mathbb{R}^d)$ provided that $u \in H^s(\mathbb{R}^d)$ for some $s > d/2$.

Exercise 2 (Operator extension | 4 points).

Let X, Z be Banach spaces and Y be a dense subset in X. Let $T : Y \longrightarrow Z$ be a bounded linear operator. Show that T is uniquely extendable to X.

Exercise 3 (Gelfand triple | 4 points).

Let V, H be Hilbert spaces and let $V \subset H$. Let the inclusion $i : V \longrightarrow H$ be continuous, injective and $\mathrm{img}(i)$ be dense in H. Furthermore, let $\|iv\|_H \le \|v\|_V$. Let the norm on V' be given in a canonical way according to

$$\|\ell\|_{V'} := \sup_{0 \neq v \in V} \frac{\ell(v)}{\|v\|_V}.$$

(a) Show that any $h \in H$ can be taken to be a continuous linear functional over V. To do this, consider $\ell_h : V \longrightarrow \mathbb{R}$ such that $\ell_h(v) := (h, iv)_H$.

(b) Conclude that $i' : H \longrightarrow V'$, $i'(h) := \ell_h$ defines an inclusion with $\|i'(h)\|_{V'} \le \|h\|_H$.

(c) Show that the inclusion i' is injective.

(d) Show that $\mathrm{img}(i')$ is dense in V'.

Hint.

(i) The image $\mathrm{img}(i')$ *is dense in* V' *if and only if there holds* $\psi = 0$ *for every* $\psi \in V''$ *satisfying* $\psi(i'h) = 0$ *for all* $h \in H$.

(ii) Since V *is a Hilbert space and thus also reflexive, then for every* $\psi \in V''$ *there is a* $v \in V$ *such that for all* $\ell \in V'$ *we have* $\psi(\ell) = \ell(v)$.

Exercise 4 (Fourier transform | 4 points).

Calculate the Fourier transform of following kernel functions:

(a) *Exponential:*

$$k_1(x) = \exp(-|x|)$$

(b) *Gaussian:*

$$k_2(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$

(c) *Matérn-3/2:*

$$k_3(x) = (1 + \sqrt{3}|x|) \exp(-\sqrt{3}|x|)$$