Exercise Sheet 1.

Hand in due to: **Friday, 10/03/2023, 14:00**

Exercise 1 (Sobolev space $H^1(\mathbb{R})$ | 4 points).

Let $f \in C^1(\mathbb{R})$ be such that $f(0) = 0$. Show that if $u \in H^1(\mathbb{R})$, then $f(u) \in H^1(\mathbb{R})$.

Exercise 2 (Sobolev spaces with real exponents | 4 points).

Let $s \in \mathbb{R}_+$ and $\Omega = (-1, 1)$. The function is given

$$u(x) = \begin{cases} -1, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

Show that $\|u\|_{W^{s,2}(\Omega)} < \infty$ holds when $0 \leq s < 1/2$.

Exercise 3 (Regularization | 4 points).

For $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, for $\alpha > 0$ and $\mathbf{x} \in \mathbb{R}^n$, let the quadratic Tikhonov functional be

$$T_\alpha(\mathbf{x}) := \|\mathbf{b} - \mathbf{A}\mathbf{x}\|^2 + \alpha\|\mathbf{x}\|^2.$$

Show:

(a) For every $\alpha > 0$ there exists a uniquely determined minimum $\mathbf{x}_\alpha \in \mathbb{R}^n$ of T_α, i.e.,

$$T_\alpha(\mathbf{x}_\alpha) \leq T_\alpha(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

(b) The minimum $\mathbf{x}_\alpha$ satisfies the linear equation system

$$(\mathbf{A}^\top\mathbf{A} + \alpha\mathbf{I})\,\mathbf{x}_\alpha = \mathbf{A}^\top\mathbf{b}.$$

Exercise 4 (Fourier transform | 4 points).

For the functions $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, show the following properties of the Fourier transform:

(a) *Differentiation:*

$$\partial^{\boldsymbol{\alpha}}(\mathcal{F}\varphi)(\boldsymbol{\xi}) = (-i)^{|\boldsymbol{\alpha}|}\mathcal{F}(\mathbf{x}^{\boldsymbol{\alpha}}\varphi)(\boldsymbol{\xi})$$
$$\boldsymbol{\xi}^{\boldsymbol{\alpha}}(\mathcal{F}\varphi)(\boldsymbol{\xi}) = (-i)^{|\boldsymbol{\alpha}|}\mathcal{F}(\partial^{\boldsymbol{\alpha}}\varphi)(\boldsymbol{\xi})$$

(b) *Translation by* $\mathbf{y} \in \mathbb{R}^d$*:*

$$\big(\mathcal{F}\varphi(\mathbf{x} + \mathbf{y})\big)(\boldsymbol{\xi}) = e^{i\langle \mathbf{y}, \boldsymbol{\xi}\rangle}(\mathcal{F}\varphi)(\boldsymbol{\xi})$$

(c) *Plancherel theorem:*

$$(\varphi, \psi)_{L^2(\mathbb{R}^d)} = (\mathcal{F}\varphi, \mathcal{F}\psi)_{L^2(\mathbb{R}^d)} \quad \text{and in particular} \quad \|\varphi\|^2_{L^2(\mathbb{R}^d)} = \|\mathcal{F}\varphi\|^2_{L^2(\mathbb{R}^d)}$$

(d) *Convolution:*

$$\mathcal{F}\varphi \cdot \mathcal{F}\psi = (2\pi)^{-d/2}\mathcal{F}(\varphi * \psi) \quad \text{with} \quad (\varphi * \psi)(\mathbf{x}) := \int_{\mathbb{R}^d} \varphi(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

(e) *Self-mapping:*

$$\mathcal{F}\varphi \in \mathcal{S}(\mathbb{R}^d)$$