Frames for the solution of operator equations in Hilbert spaces with fixed dual pairing

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Abstract

For the solution of operator equations, Stevenson introduced in [52] a definition of frames, where a Hilbert space and its dual are *not* identified. This means that the Riesz isomorphism is not used as an identification, that, for example, does not make sense for the Sobolev spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. In this article, we are going to revisit this concept of Stevenson frames and introduce it for Banach spaces. This is equivalent to ℓ^2-Banach frames. It is known that, if such a system exists, by defining a new inner product and using the Riesz isomorphism, the Banach space is isomorphic to a Hilbert space. In this article, we deal with the contrasting setting, where $\mathcal{H}$ and $\mathcal{H}'$ are not identified, and equivalent norms are distinguished, and show that in this setting the investigation of ℓ^2-Banach frames make sense.

Keywords: Frames, Banach frames, Stevenson frames, matrix representation, discretization of operators, invertibility.

1 Introduction

The standard definition of frames found in the first paper by Duffin and Schaefer [31] is the following:

$$\|f\|_{\mathcal{H}} \approx \|\langle f, \psi_k \rangle_{\mathcal{H}}\|_{\ell^2} \text{ for all } f \in \mathcal{H}. \tag{1}$$

Here, $x \approx y$ means that there are constants $0 < A \leq B < \infty$ such that $A \cdot x \leq y \leq B \cdot x$.

This concept led to a lot of theoretical work, see e.g. [1, 14, 23, 49, 50], but has been used also extensively in signal processing [16], quantum mechanics [33], acoustics [10] and various other fields.

Frames can be used also to represent operators. For the numerical solution of operator equations, the (Petrov-) Galerkin scheme [51] is used, where operators are represented by $\langle O\psi_k, \phi_l \rangle_{k,l \in K}$, called the *stiffness* or *system matrix*. The collection $\Psi = (\psi_k)_{k \in K}$ consists of the ansatz functions, the collection $\Phi = (\phi_k)_{k \in K}$ are the test functions. If Ψ and Φ live in the same space, this is called Galerkin scheme, otherwise it is called Petrov-Galerkin scheme.

In finite and boundary element approaches, bases were used [29, 32], but also frames have been applied, e.g. in [36, 39, 40, 47, 52]. Recently, such operator representation got also a more theoretical treatment [6, 9, 12].

In numerical applications, it is often advantageous to have self-adjoint matrices, e.g. for Krylov subspace methods, which necessitates to use the same sequence for the discretization at both sides, i.e. investigating $\langle O\psi_k, \psi_l \rangle_{k,l \in K}$. Note that this matrix is self-adjoint if O is, and semi-positive if O is. Positivity is in general not preserved, only if a system without redundancy is used, i.e. a Riesz sequence. Partial differential operators are typically operators of the form $O : \mathcal{H} \to \mathcal{H}'$, while boundary integral operators might also be smoothing operators which map in accordance with $O : \mathcal{H}' \to \mathcal{H}$. One possible solution is to work with Gelfand triples i.e, $\mathcal{H} \subseteq \mathcal{H}_0 \subseteq \mathcal{H}'$. This is explicitly done for the concept of Gelfand frames [27].

Another possibility is the following, introduced by Stevenson in [52] and used e.g. in [40]: A collection $\Psi = (\psi_k)_{k \in K} \subset \mathcal{H}$ is called a (Stevenson) frame for $\mathcal{H}$, if

$$\|f\|_{\mathcal{H}'} \approx \left\| \langle f, \psi_k \rangle_{\mathcal{H}',\mathcal{H}} \right\|_{\ell^2} \quad \text{for all } f \in \mathcal{H}'. \tag{2}$$

Note the difference to the definition (1) by Duffin and Schaefer, which is significant only if the Riesz isomorphism is not employed. Here, the Gelfand triple is only implicitly used and, if the fully general setting is used, the density of the spaces is not required.

Clearly, the definitions 2 and 1 are equivalent by the Riesz isomorphism. On the other hand, if the isomorphism $\mathcal{H} \cong \mathcal{H}'$ is not considered, but another one is utilized, for example, considering the triple $\mathcal{H} \subset \mathcal{H}_0 \subset \mathcal{H}'$, then the Riesz isomorphism is usually used as an identification on the pivot space $\mathcal{H}_0 \cong \mathcal{H}_0'$, and therefore $\mathcal{H}$ and $\mathcal{H}'$ cannot be considered to be equal.

In this article, we consider the original definition by Stevenson and re-investigate in full detail all the derivation to ensure that the Riesz identifi-

cation does not 'creep in' again.

On a more theoretical level, let us consider Banach frames [21, 24, 37]. Thus, we consider a Banach space X, a sequence space X_d, and a sequence $\Psi \subset X'$. This is a X_d-frame if

$$\|f\|_X \approx \left\| \langle f, \psi_k \rangle_{X,X'} \right\|_{X_d} \quad \text{for all } f \in X.$$

It is called a Banach frame if a reconstruction operator exists, i.e. there exists $R : X_d \to X$ with $R(\psi_k(f)) = f$ for all $f \in X$.

In this setting, ℓ_2-frames were not considered to be interesting as they are isomorph to Hilbert frames, see e.g. [54, Proposition 3.10]: Let Ψ be a ℓ^2-frame for X. Then, X can be equipped with an inner product $\langle f, g \rangle_X = \langle C_\Psi f, C_\Psi g \rangle_{\ell^2}$, becoming a Hilbert space, and Ψ is a (Hilbert) frame for X. The proof uses the Riesz isomorphism $\mathcal{H} \cong \mathcal{H}'$ in the last line. But if a context is considered, where this isomorphism cannot be applied, like for example a Gelfand triple setting, suddenly the concept of ℓ_2-frames might become non-trivial again, and the concept of Stevenson frames is different to a frame. In this article, we investigate this approach.

The rest of this article is structured as follows. In Section 2, we motivate Gelfand triples $\mathcal{H}' \subset \mathcal{H}_0 \subset \mathcal{H}$ by a simple example arising from the variational formulation of second order elliptic partial differential equations. Section 3 then provides the main ingredients we need, especially it introduces the different notions of frames for solving operator equations. By an illustrative example, we show that Stevenson frames seem to offer the most flexible concept for the discretization of operator equations. Finally, in Section 4, we generalize Stevenson frames to Banach spaces and discuss the consequences.

2 Motivation: Solving Operator Equations

Let $O : \mathcal{H} \to \mathcal{H}'$ and define the bilinear form $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by $a(u,v) = \langle Ou, v \rangle$. Assume that a satisfies the following properties:

1. Let a be bounded, i.e. there is a constant C_S, such that

$$a(u,v) \leq C_S \cdot \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} .$$

   This is equivalent to O being bounded.

2. Let a be elliptic, i.e. there exists a constant C_E such that

$$a(u,u) \geq C_E \cdot \|u\|_{\mathcal{H}}^2 .$$

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Both conditions are equivalent to O being bounded, boundedly invertible, and positive, see e.g. [18, 20].

The general goal is to find the solution $u \in \mathcal{H}$ such that

$$a(u,v) = \ell(v) \text{ for all } v \in \mathcal{H}. \tag{3}$$

This is the weak formulation of the operator equation $Ou = b$, setting $\ell(v) = \langle b, v \rangle_{\mathcal{H}',\mathcal{H}}$ for $u \in \mathcal{H}$ and $b \in \mathcal{H}'$.

In numerical approximation schemes, to get an approximate solution, finite dimensional subspaces $V \subset \mathcal{H}$ are considered and the solution $u_V \in V$ such that

$$a(u_V, v) = \ell(v) \text{ for all } v \in V$$

is calculated. The error between the continuous solution $u \in \mathcal{H}$ and the approximate solution $u_V \in V$ is orthogonal to the space V, which is known as the Galerkin orthogonality: $a(u - u_V, v) = 0$ for all $v \in V$. Note that, in difference to e.g. a Gelfand triple approach, the norms on V and $\mathcal{H}$ are the same in the setting above. Still, we are usually in a Gelfand triple setting here: $\mathcal{H} \subset \mathcal{H}_0 \subset \mathcal{H}'$.

We shall illustrate the setting also by a practical example from the theory of partial differential equations. To that end, assume that Ω is a bounded domain in $\mathbb{R}^d$ and let $\mathcal{H}_0 := L^2(\Omega)$ be the space of all square-integrable functions $v : \Omega \to \mathbb{R}$. As space $\mathcal{H} \subset \mathcal{H}_0$ we consider the Sobolov space $H_0^1(\Omega)$ which consists of all functions in $L^2(\Omega)$ whose first order week derivatives are also square-integrable and which are zero at the boundary $\partial\Omega$. Thus, the variational formulation of the Poisson equation

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

reads

$$\text{seek } u \in H_0^1(\Omega) \text{ such that } a(u,v) = l(v) \text{ for all } v \in H_0^1(\Omega), \tag{4}$$

compare [18] for example. The bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}, \quad a(u,v) = \int_\Omega \nabla u \nabla v \, \mathrm{d}\mathbf{x}$$

is continuous and elliptic due to Friedrichs' inequality, cf. [18], and the linear form

$$\ell : H_0^1(\Omega) \to \mathbb{R}, \quad \ell(v) = \int_\Omega f v \, \mathrm{d}\mathbf{x}$$

is continuous provided that $f \in H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)'$. Hereby, the inner product in the pivot space $L^2(\Omega)$ is continuously extended onto the duality pairing $H^{-1}(\Omega) \times H_0^1(\Omega)$. Hence, the underlying Gelfant triple is $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$.

3 Main Definitions and Notations

3.1 Dual Pairs

Let X, Y be vector spaces and $a(x, y)$ a bilinear functional on $X \times Y$. Then (X, Y) is called a dual pair [55], if

1. $\forall x \in X \backslash \{0\} \ \exists y \in Y$ s.t. $a(x, y) \neq 0$,

2. $\forall y \in Y \backslash \{0\} \ \exists x \in X$ s.t. $a(x, y) \neq 0$.

In short, the notation $a(x, y) = \langle x, y \rangle_a = \langle x, y \rangle$ is used. A classical example is a Banach space X and its dual space X'. But looking at other dual pairs allows to have an explicit form for the dual elements [41].

Note that often an isomorphism is considered as an identity. For example, by using the Riesz mapping $\mathcal{H} \cong \mathcal{H}'$, the dual space $\mathcal{H}'$ is often identified with $\mathcal{H}$. If two or more isomorphisms are involved, this identification, of course, can only be considered for one of those isomorphisms. For example, if we consider two Hilbert space $\mathcal{H}_1 \subset \mathcal{H}_2$, the Riesz isomorphism can be considered only for one of them to be an identification, see also Section 3.3.2.

3.2 Gelfand triples

Let X be a Banach space and $\mathcal{H}$ a Hilbert space. Then, the triple $(X, \mathcal{H}, X')$ is called a Banach Gelfand triple [25], if $X \subset \mathcal{H} \subset X'$, where X is dense in $\mathcal{H}$, and $\mathcal{H}$ is $w^\star$-dense in X'. The prototype of such a triple is $(\ell^1, \ell^2, \ell^\infty)$ in case of sequence spaces.

Note that, even if we consider the spaces all being Hilbert spaces such a sequences is also called rigged Hilbert spaces [2], the Riesz isomorphism, in general, is not just the composition of the inclusion with its adjoint. This depends on the chosen concrete dual pairing.

As another example, consider the triple $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$, which has been presented in the practical example for the Poisson equation in Section 2.

3.3 Frames

A sequence $\Psi = (\psi_k)_{k \in K}$ in a separable Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$, if there exist positive constants A_Ψ and B_Ψ (called lower and upper frame bound, respectively) that satisfy

$$A_\Psi \|f\|^2 \leq \sum_{k \in K} |\langle f, \psi_k \rangle|^2 \leq B_\Psi \|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{5}$$

An upper (resp. lower) semi-frame is a complete system that only satisfies the upper (resp. lower) frame inequality, see [4, 3]. A frame where the two bounds can be chosen to be equal, i.e. $A_\Psi = B_\Psi$, is called *tight*. We will denote the corresponding sequences in $\mathcal{H}$ by $\Psi = (\psi_k)$ and $\Phi = (\phi_k)$ in the following. We consider general discrete index sets $K \subset \mathbb{R}^d$. A sequence that is a frame for its closed linear span is called a frame sequence.

By $C_\Psi : \mathcal{H} \to \ell^2$ we denote the *analysis operator* defined by $(C_\Psi f)_k = \langle f, \psi_k \rangle$. The adjoint of C_Ψ is the *synthesis operator* $D_\Psi(c_k) = \sum_k c_k \psi_k$. The *frame operator* $S_\Psi = D_\Psi C_\Psi$ can be written as $S_\Psi f = \sum_k \langle f, \psi_k \rangle \psi_k$. It is positive and invertible. Note that those 'frame-related' operators can be defined as possibly unbounded operators for any sequence in the Hilbert space [13].

By using the *canonical dual frame* $(\tilde{\psi}_k)$, i.e., $\tilde{\psi}_k = S_\Psi^{-1} \psi_k$ for all k, we get a reconstruction formula:

$$f = \sum_k \langle f, \psi_k \rangle \tilde{\psi}_k = \sum_k \left\langle f, \tilde{\psi}_k \right\rangle \psi_k \text{ for all } f \in \mathcal{H}.$$

The *Gramian matrix* G_Ψ is defined by $(G_\Psi)_{k,l} = \langle \psi_l, \psi_k \rangle$, also called the mass matrix. This matrix defines an operator on ℓ^2 by matrix multiplication, corresponding to $G_\Psi = C_\Psi D_\Psi$. Similarily, we can define the *cross-Gramian matrix* $(G_{\Psi,\Phi})_{k,l} = \langle \phi_l, \psi_k \rangle$ between two different frames Φ and Ψ. Clearly,

$$G_{\Psi,\Phi} c = \sum_l (G_{\Psi,\Phi})_{k,l} c_l = \left\langle \sum_l c_l \phi_l, \psi_k \right\rangle = C_\Psi D_\Phi c.$$

If, for the sequence Ψ, there exist constants $A_\Psi, B_\Psi > 0$ such that the inequalities

$$A_\Psi \|c\|_2^2 \leq \left\| \sum_{k \in K} c_k \psi_k \right\|_{\mathcal{H}}^2 \leq B_\Psi \|c\|_2^2$$

are fulfilled, Ψ is called a *Riesz sequence*. If Ψ is complete, it is called a *Riesz basis*.

3.3.1 Banach Frames

The concept of frames can be extended to Banach spaces [37, 21, 24]:

Let X be a Banach space and X_d be a Banach space of scalar sequences. A sequence (ψ_k) in the dual $X^\star$ is called a X_d-frame for the Banach space X ($1 < p < \infty$), if there exist constants $A_\Psi, B_\Psi > 0$ such that

$$A_\Psi \|f\|_X \leq \|\psi_k(f)\|_{X_d} \leq B_\Psi \|f\|_X \text{ for all } f \in X.$$

An X_d-frame is called a Banach frame with respect to a sequence space X_d, if there exists a bounded reconstruction operator $R : X_d \to X$, such that $R(\psi_k(f)) = f$ for all $f \in X$. In our setting, we use p-frames, that is $X_d = \ell^p$ for $1 \leq p \leq \infty$, especially, we use $X_d = \ell^2$.

A family $(g_k)_{k \in K} \subset X$ is called a *q-Riesz sequence* $(1 \leq q \leq \infty)$ for X, if there exist constants $A_\Psi, B_\Psi > 0$ such that

$$A_\Psi \left(\sum_{k \in K} |d_k|^q \right)^{\frac{1}{q}} \leq \left\| \sum_{k \in K} d_k g_k \right\|_X \leq B_\Psi \left(\sum_{k \in K} |d_k|^q \right)^{\frac{1}{q}} \tag{6}$$

for all finite scalar sequence (d_k). The family is called a q-Riesz basis if it fulfills (6) and $\overline{\text{span}}\{g_k\}_{k \in K} = X$.

Any q-Riesz basis for $X^\star$ is a p-frame for X, where $\frac{1}{p} + \frac{1}{q} = 1$, compare [24].

3.3.2 Gelfand Frames

A frame for $\mathcal{H}$ is called a *Gelfand frame* [27] for the Gelfand triple $(X, \mathcal{H}, X')$ if there exists a Gelfand triple of sequence spaces (X_d, ℓ^2, X_d'), such that the synthesis operator $D_\Psi : X_d \to X$ and the analysis operator $C_{\widetilde{\Psi}} : X \to X_d$ are bounded. As a result, see [27, 44], this means that Ψ is a Banach frame for X_d and $\widetilde{\Psi}$ a Banach frame for X_d'.

In many approaches, see e.g. [27], it is assumed for the implementation that there exists an isomorphism $D_B : X_d \to \ell^2$. Should X_d be non-reflexive, then it is also assumed that $D_B^\star$ is an isomorphism. If D_B is a diagonal operator, i.e. $D_B = \text{diag}(w_k)$ and $D_B^{-1} = \text{diag}\left(\frac{1}{w_k}\right)$, then $\Psi = \left(\frac{1}{w_k}\psi_k\right)$ is a Hilbert frame for X and $\left(w_k\widetilde{\psi}_k\right)$ is a Hilbert frame for X'. This is shown for real weights in [56]. It is easy to see also for complex weights when using a weighted frame viewpoint [7, 53]. These cases cover the weighted spaces ℓ^2_w.

The above setting can be generalized as follows: We define, similar to [54], the sesqui-linear form $\langle f, g \rangle^o_X := \left\langle D_B C_{\widetilde{\Psi}} f, D_B C_{\widetilde{\Psi}} g \right\rangle_{\ell^2}$. It is obviously bounded and coercive, and, in particular, $\|f\|_{X^o} := \sqrt{\langle f, f \rangle_{X^o}}$ is equivalent to $\|f\|_X$. Therefore, $(X, \|f\|_{X^o})$ is a Hilbert space which is isomorphic to $(X, \|f\|_X)$. Now let $\xi_l := D_\Psi D_B^{-1} \delta_l$, where δ_l is the standard basis in ℓ^2. This is a Hilbert space frame for X. Similarly, $\eta_l := D_{\widetilde{\Psi}} \left(D_B^\star\right)^{-1} \delta_l$ is a Hilbert space frame for X'. As a consequence X and X' are Hilbert spaces, but $X \neq X'$ and the inner products and the corresponding norms are changed, albeit equivalent to the original ones.

3.4 Stevenson Frames

We consider the duality $(\mathcal{H}, \mathcal{H}')$ without using the Riesz isomorphism. In particular, we use the duality with respect to a second Hilbert space $\mathcal{H}_0$.

Definition 3.1 ([52]) *A sequence $\Psi = (\psi_k)_{k \in K} \subset \mathcal{H}$ is called a (Stevenson) $\mathcal{H}'$-frame if there exists constants $0 < A_\Psi \le B_\Psi < \infty$ such that*

$$A_\Psi \cdot \|f\|_{\mathcal{H}'}^2 \le \left\| \langle f, \psi_k \rangle_{\mathcal{H}',\mathcal{H}} \right\|_{\ell^2} \le B_\Psi \cdot \|f\|_{\mathcal{H}'}^2 \ \text{ for all } f \in \mathcal{H}. \tag{7}$$

Different to the Gelfand frames setting, we do not assume density.

Typically, we consider Sobolev spaces and the L^2-inner product, which we can consider as co-orbit spaces with the sequence spaces ℓ_w^2 varying w. Here, invertible operator between different spaces exist, see Section 3.3.2, and density is also given. In this article, we treat the most general setting.

In [52], the author states "We adapted the definition of a frame given in [30, Section 3] by identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ via the Riesz mapping". Then, the following results are stated, also in [40], without proofs: The analysis operator $C_\Psi : \mathcal{H}' \to \ell^2$, $C_\Psi(f) = (\langle f, \psi_k \rangle)_{k \in K}$ is bounded by (7), as is its adjoint $C_\Psi^\star : \ell^2 \to \mathcal{H}$. It can be easily shown that $C_\Psi^\star = D_\Psi$ is the synthesis operator with $D_\Psi c = \sum_{k \in K} c_k \psi_k$. Especially, one has

$$\ell^2 = \operatorname{ran}(C_\Psi) \oplus \ker(D_\Psi).$$

Define the frame operator $S_\Psi = D_\Psi C_\Psi$. It is a mapping $S_\Psi : \mathcal{H}' \to \mathcal{H}$, which is boundedly invertible. We can show that the sequence $\widetilde{\Psi} = \left(S_\Psi^{-1} \psi_k\right)_{k \in K}$ is a (Stevenson) $\mathcal{H}$-frame with bounds $\frac{1}{B_\Psi}$ and $\frac{1}{A_\Psi}$. Here, $C_{\widetilde{\Psi}} = C_\Psi S_\Psi^{-1}$ and $D_{\widetilde{\Psi}} = S_\Psi^{-1} D_\Psi$. Furthermore, it holds $S_{\widetilde{\Psi}} = S_\Psi^{-1}$ and, therefore, $S_{\widetilde{\Psi}} : \mathcal{H} \to \mathcal{H}'$.

We have the reconstructions

$$f = D_\Psi C_{\widetilde{\Psi}} h = \sum_{k \in K} \left\langle f, \tilde{\psi}_k \right\rangle_{\mathcal{H},\mathcal{H}'} \psi_k, \tag{8}$$

and

$$h = D_{\widetilde{\Psi}} C_\Psi h = \sum_{k \in K} \langle h, \psi_k \rangle_{\mathcal{H}',\mathcal{H}} \, \tilde{\psi}_k, \tag{9}$$

for all $f \in \mathcal{H}$ and $h \in \mathcal{H}'$.

The cross-Gramian matrix $G_{\Psi,\widetilde{\Psi}} = D_\Psi C_{\widetilde{\Psi}}$ is the orthogonal projection on $\operatorname{ran}(C_\Psi)$ and coincides with $G_{\widetilde{\Psi},\Psi}$. Therefore, $\operatorname{ran}(C_\Psi) = \operatorname{ran}(C_{\widetilde{\Psi}})$.

In this article, we are revisiting those statements, make them slightly more general, in order to make sure that the Riesz isomorphism was not used.

3.5 An Illustrative Example

Let $\Omega \subset \mathbb{R}^n$ be a sufficiently smooth, bounded domain. We consider a multi-scale analysis, i.e., a dense, nested sequence of finite dimensional subspaces

$$V_0 \subset V_1 \subset \cdots \subset V_j \subset \cdots \subset L^2(\Omega),$$

consisting of piecewise polynomial ansatz functions $V_j = \text{span}\{\varphi_{j,k} : k \in \Delta_j\}$, such that $\dim V_j \sim 2^{jn}$ and

$$L^2(\Omega) = \overline{\bigcup_{j \in \mathbb{N}_0} V_j}, \qquad V_0 = \bigcap_{j \in \mathbb{N}_0} V_j.$$

One might think here of a multigrid decomposition of standard Lagrangian finite element spaces or of a sequence of spline spaces originating from dyadic subdivision.

Trial spaces V_j which are used for the Galerkin method satisfy typically a *direct* or *Jackson* estimate. This means that

$$\|v - P_j v\|_{L^2(\Omega)} \leq C_J 2^{-jq} \|v\|_{H^q(\Omega)}, \quad v \in H^q(\Omega), \tag{10}$$

holds for all $0 \leq q \leq d$ uniformly in j. Here, $P_j : L^2(\Omega) \to V_j$ is the $L^2(\Omega)$-orthogonal projection onto the trial space V_j and $H^q(\Omega) \subset L^2(\Omega)$, $q \geq 0$, denotes the Sobolev space of order q. The upper bound $d > 0$ refers in general to the maximum order of the polynomials which can be represented in V_j, while the factor 2^{-j} refers to the mesh size of V_j, i.e., the diameter of the finite elements, compare [18] for example.

Besides the Jackson type estimate (10), there also holds the *inverse* or *Bernstein* estimate

$$\left\| P_j v \right\|_{H^q(\Omega)} \leq C_B 2^{jq} \left\| P_j v \right\|_{L^2(\Omega)}, \quad v \in H^q(\Omega), \tag{11}$$

for all $0 \leq q < \gamma$, where the upper bound

$$\gamma := \sup \left\{ t \in \mathbb{R} : V_j \subset H^t(\Omega) \right\} > 0$$

refers to the regularity of the functions in V_J. There holds $\gamma = d - 1/2$ for trial functions based on cardinal B-splines, since they are globally C^{d-1}-smooth, and $\gamma = 3/2$ for standard Lagrangian finite element shape functions, since they are only globally continuous.

A crucial requirement is the uniform frame stability of the systems under consideration, i.e., the existence of constants $A_\Phi, B_\Phi > 0$ such that

$$A_\Phi \left\| P_j f \right\|_{L^2(\Omega)}^2 \leq \sum_{k \in \Delta_j} |\langle f, \varphi_{j,k} \rangle|^2 \leq B_\Phi \left\| P_j f \right\|_{L^2(\Omega)}^2 \text{ for all } f \in V_j \tag{12}$$

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holds uniformly for all j. This stability is satisfied for example by Lagrangian finite element basis functions defined on a multigrid hierarchy resulting from uniform refinement of a given coarse grid, see [18] for example. It is also satisfied by B-splines defined on a dyadic subdivision of the domain under consideration.

Having a multiscale analysis at hand, it can be used for telescoping a given function to account for the fact that Sobolev norms act different on different length scales. Namely, the interplay of (10) and (11) gives rise to the norm equivalence

$$\|f\|_{\widetilde{H}^{-q}(\Omega)}^2 \sim \sum_{j\in\mathbb{N}_0} 2^{-2jq}\big\|(P_j - P_{j-1})f\big\|_{L^2(\Omega)}^2 \tag{13}$$

for all $0 \le q < \gamma$, where $P_{-1} := 0$ and $\widetilde{H}^{-q}(\Omega) := \big(H^q(\Omega)\big)'$ denotes the dual to $H^q(\Omega)$, see [28] for a proof.

In accordance with [40], using (12), we can estimate

$$\sum_{j\in\mathbb{N}_0}\sum_{k\in\Delta_j} 2^{-2jq}|\langle f, \varphi_{j,k}\rangle|^2 \approx \sum_{j\in\mathbb{N}_0} 2^{-2jq}\big\|P_j f\big\|_{L^2(\Omega)}^2$$

$$= \sum_{j\in\mathbb{N}_0} 2^{-2jq}\sum_{\ell=0}^{j}\big\|(P_\ell - P_{\ell-1})f\big\|_{L^2(\Omega)}^2$$

$$= \sum_{\ell\in\mathbb{N}_0}\big\|(P_\ell - P_{\ell-1})f\big\|_{L^2(\Omega)}^2 \sum_{j=\ell}^{\infty} 2^{-2jq}.$$

The latter sum converges provided that $q > 0$ and we arrive at

$$\sum_{j\in\mathbb{N}_0}\sum_{k\in\Delta_j} 2^{-2jq}|\langle f, \varphi_{j,k}\rangle|^2 \approx \sum_{\ell\in\mathbb{N}_0} 2^{-2\ell q}\big\|(P_\ell - P_{\ell-1})f\big\|_{L^2(\Omega)}^2.$$

In view of the norm equivalence (13), we have thus proven that there exist constants $A_\Phi, B_\Phi > 0$ such that

$$A_\Phi\|f\|_{\widetilde{H}^{-q}(\Omega)}^2 \le \sum_{j\in\mathbb{N}_0}\sum_{k\in\Delta_j} 2^{-2jq}|\langle f, \varphi_{j,k}\rangle|^2 \le B_\Phi\|f\|_{\widetilde{H}^{-q}(\Omega)}^2 \tag{14}$$

for all $0 < q < \gamma$. Therefore, in accordance with Definition 3.1, the collection

$$\Phi = \{2^{-jq}\varphi_{j,k} : k \in \Delta_j,\, j \in \mathbb{N}_0\} \tag{15}$$

defines a Stevenson frame for $\mathcal{H} = H^q(\Omega)$, where $\mathcal{H}' = \widetilde{H}^{-q}(\Omega)$ with duality related to $\mathcal{H}_0 = L^2(\Omega)$. Notice that this frame underlies the construction of

the so-called BPX preconditioner, see e.g. [19, 28, 46]. Especially, by removing all basis functions which are associated with boundary nodes, one gets a Stevenson frame for $\mathcal{H} = H_0^1(\Omega)$, as required for the Galerkin discretization of elliptic partial differential equations, compare Section 2.

We like to emphazise that the collection (15) does not define a Gelfand frame, since (14) does not hold in the $\mathcal{H}_0 = L^2(\Omega)$, i.e., for $q = 0$. Hence, the concept of Stevenson frames seems to be more flexible than the concept of Gelfand frames.

3.6   Operator Representation in Frame Coordinates

For orthonormal sequences, it is well known that operators can be uniquely described by a matrix representation [35]. The same can be constructed with frames and their duals, see [6, 9].

Let $\Psi = (\psi_k)$ be a frame in $\mathcal{H}_1$ with bounds $A_\Psi, B_\Psi > 0$, and let $\Phi = (\phi_k)$ be a frame in $\mathcal{H}_2$ with $A_\Phi, B_\Phi > 0$.

1. Let $O : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded, linear operator. Thus, the infinite matrix

$$\left(\mathcal{M}^{(\Phi,\Psi)}(O) \right)_{m,n} = \langle O\psi_n, \phi_m \rangle$$

   defines a bounded operator from ℓ^2 to ℓ^2 with $\|\mathcal{M}\|_{\ell^2 \to \ell^2} \leq \sqrt{B_\Phi \cdot B_\Psi} \cdot \|O\|_{\mathcal{H}_1 \to \mathcal{H}_2}$. As an operator $\ell^2 \to \ell^2$, we have

$$\mathcal{M}^{(\Phi,\Psi)}(O) = C_\Phi \circ O \circ D_\Psi.$$

2. On the other hand, let M be an infinite matrix defining a bounded operator from ℓ^2 to ℓ^2, $(Mc)_i = \sum_k M_{i,k} c_k$. Then, the operator $\mathcal{O}^{(\Phi,\Psi)}$ defined by

$$\left(\mathcal{O}^{(\Phi,\Psi)}(M) \right) h = \sum_k \left(\sum_j M_{k,j} \langle h, \psi_j \rangle \right) \phi_k \text{ for all } h \in \mathcal{H}_1$$

   is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ with

$$\left\| \mathcal{O}^{(\Phi,\Psi)}(M) \right\|_{\mathcal{H}_1 \to \mathcal{H}_2} \leq \sqrt{B_\Phi \cdot B_\Psi} \, \|M\|_{\ell^2 \to \ell^2}$$

   and

$$\mathcal{O}^{(\Phi,\Psi)}(M) = D_\Phi \circ M \circ C_\Psi = \sum_k \sum_j M_{k,j} \cdot \phi_k \otimes_i \psi_j.$$

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Please note that there is a classification of matrices that are bounded operators from ℓ^2 to ℓ^2 [26].

If we start out with frames, more properties can be proved [6]: Let $\Psi = (\psi_k)$ be a frame in $\mathcal{H}_1$ with bounds $A_\Psi, B_\Psi > 0$, $\Phi = (\phi_k)$ in $\mathcal{H}_2$ with $A_\Phi, B_\Phi > 0$.

1. It holds
$$\left(\mathcal{O}^{(\Phi,\Psi)} \circ M^{(\tilde{\Phi},\tilde{\Psi})} \right) = \mathsf{id}_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} = \left(\mathcal{O}^{(\tilde{\Phi},\tilde{\Psi})} \circ M^{(\Phi,\Psi)} \right).$$

Therefore, for all $O \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$:
$$O = \sum_{k,j} \left\langle O\tilde{\psi}_j, \tilde{\phi}_k \right\rangle \phi_k \otimes_i \psi_j.$$

2. $\mathcal{M}^{(\Phi,\Psi)}$ is injective and $\mathcal{O}^{(\Phi,\Psi)}$ is surjective.

3. If $\mathcal{H}_1 = \mathcal{H}_2$, then $\mathcal{O}^{(\Psi,\tilde{\Psi})}(\mathsf{id}_{\ell^2}) = \mathsf{id}_{\mathcal{H}_1}$.

4. Let $\Xi = (\xi_k)$ be any frame in $\mathcal{H}_3$, and $O : \mathcal{H}_3 \to \mathcal{H}_2$ and $P : \mathcal{H}_1 \to \mathcal{H}_3$. Then, it holds
$$\mathcal{M}^{(\Phi,\Psi)}(O \circ P) = \left(\mathcal{M}^{(\Phi,\Xi)}(O) \cdot \mathcal{M}^{(\tilde{\Xi},\Psi)}(P) \right).$$

Note that, in the Hilbert space of Hilbert-Schmidt operators, the tensor product $\Psi \otimes \Phi := \{\psi_k \otimes \psi_l\}_{(k,l) \in K \times K}$ is a Bessel sequence / frame sequence / Riesz sequence, if the starting sequences Ψ and Φ are [5], with $\mathcal{M}^{(\Phi,\Psi)}$ being the analysis and $\mathcal{O}^{(\Phi,\Psi)}$ being the synthesis operator. This relation is even an equivalence [17].

For the invertibility, it can be shown [11, 12]: If and only if O is bijective, then $M = \mathcal{M}^{(\Phi,\Psi)}(O)$ is bijective as operator from $\mathrm{ran}\,(C_\Psi)$ onto $\mathrm{ran}\,(C_\Phi)$. In this case, one has
$$M^\dagger = \mathcal{M}^{(\tilde{\Psi},\tilde{\Phi})}(O^{-1}) = G_{\tilde{\Psi},\tilde{\Phi}} \circ \mathcal{M}^{(\Phi,\Psi)}\left(O^{-1}\right) G_{\tilde{\Psi},\tilde{\Phi}} = \mathcal{M}^{(\Psi,\Phi)}\left(S_\Psi^{-1} O^{-1} S_\Phi^{-1}\right).$$

If we have an operator equation $Ou = b$, we use
$$Ou = b \iff \sum_k \left\langle u, \tilde{\psi}_k \right\rangle O\psi_k = b,$$

which implies
$$\sum_k \left\langle u, \tilde{\psi}_k \right\rangle \langle O\psi_k, \psi_l \rangle = \langle b, \psi_l \rangle$$

for all $l \in K$. Setting $\mathbf{M} = \mathcal{M}^{(\Psi,\Psi)}(O)$, $\vec{u} = C_{\widetilde{\Psi}}u$ and $\vec{b} = C_{\Psi}b$, we thus have

$$Ou = b \Longleftrightarrow \mathbf{M}\vec{u} = \vec{b}.$$

Note that, for numerical computations, see e.g. [27, 52] the system of linear equations $\mathbf{M}\vec{u} = \vec{b}$ is solved. Then, $u = D_{\Psi}\vec{u}$ is the solution to $Ou = b$, avoiding the numerically expensive calculation of a dual frame [8, 43, 48]. If the frame is redundant, then u_k can be different to $\langle u, \tilde{\psi}_k \rangle$. If a Tychonov regularization is used, we obtain $u_k = \langle u, \tilde{\psi}_k \rangle$ by [38, Prop. 5.1.4].

4 Stevenson Frames Revisited

As some of the references dealing with Stevenson frames used an unlucky formulation, when stating if or if not the Riesz isomorphism is used, see e.g. [27, 52], the authors decided to check everything again, and pay particular attention to the avoidance of the Riesz isomorphism, i.e. to not use $\mathcal{H} \cong \mathcal{H}'$.

To *not* use the Riesz isomorphism in a treatment of Hilbert spaces is mind-boggling, so we decided to use Banach spaces, to be sure to avoid all pitfalls. (Note, however, that the Riesz isomorphism will be used on the sequence space ℓ^2.) In particular, this is a generalization of the original definition. The used spaces are necessarily isomorphic to Hilbert spaces, but not Hilbert spaces per se.

4.1 Stevenson Banach Frames

We start out with a generalized definition. (We will show that this is isomorphic, but *not* identical to the original definition.)

Definition 4.1 *Let (X, X') be a dual pair of reflexive Banach spaces. Let $\Psi = (\psi_k)_{k \in K} \subseteq X$. It is called a Stevenson Banach frame for X', if there exist bounds $0 < A_{\Psi} \leq B_{\Psi} < \infty$ such that*

$$A_{\Psi} \|f\|_{X'}^2 \leq \left\| \langle f, \psi_k \rangle_{X',X} \right\|_{\ell^2} \leq B_{\Psi} \|f\|_{X'}^2 \text{ for all } f \in X'.$$

The analysis operator

$$C_{\Psi} : X' \to \ell^2, \quad C_{\Psi}(f) = \left(\langle f, \psi_k \rangle_{X',X} \right)_{k \in K}$$

is bounded by $\sqrt{B_{\Psi}}$ by definition. (Note that we use here the notation which is more common for Banach spaces [37].) As a consequence of the open mapping theorem, C_{Ψ} is one-to-one and has closed range.

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For $d = (d_k) \in \ell^2(K)$ with finitely many non-zero entries, i.e. $d \in c^{00}$, consider

$$\langle C_\Psi f, d \rangle_{\ell^2} = \sum_{k \in K} \langle f, \psi_k \rangle_{X',X} \, d_k = \left\langle f, \sum_{k \in K} d_k \psi_k \right\rangle_{X',X}.$$

By using a standard density argument and the reflexivity, it can easily be shown that $C_\Psi^\star = D_\Psi$, where $D_\Psi : \ell^2 \to X$ is the synthesis operator with $D_\Psi c = \sum_{k \in K} c_k \psi_k$. The bound of D_Ψ is also $\sqrt{B_\Psi}$. The sum converges unconditionally. Indeed, consider $c \in \ell^2$. Then, let $K_0 \subset K$ be a finite set, such that

$$\sum_{k \notin K_0} |c_k|^2 < \epsilon' := \frac{\epsilon}{\sqrt{B_\Psi}}.$$

For another finite index set $K_1 \supseteq K_0$, we thus find

$$\left\| \sum_{k \notin K} c_k \psi_k - \sum_{k \notin K_1} c_k \psi_k \right\|_\mathcal{H} = \|D_\Psi (c - c \cdot \chi_{K_1})\|_\mathcal{H} < \sqrt{B_\Psi} \epsilon' = \epsilon.$$

Hence, by e.g. [55, IV.5.1] and the fact that ℓ^2 is a Hilbert space, we deduce

$$\ell^2 = \mathrm{ran}\,(C_\Psi) \oplus \ker\,(D_\Psi). \tag{16}$$

We define the frame operator $S_\Psi = D_\Psi C_\Psi$, which is a mapping $S_\Psi : X' \to X$. In particular, the operator S_Ψ is self-adjoint. By definition of S_Ψ, it follows that

$$\langle S_\Psi f, g \rangle_{X,X'} \leq \|S_\Psi f\|_X \|g\|_{X'} \leq B_\Psi \cdot \|f\|_{X'} \cdot \|g\|_{X'}. \tag{17}$$

Hence, S_Ψ is bounded with bound B_Ψ. Furthermore, we have

$$\langle S_\Psi f, f \rangle_{X,X'} = \langle C_\Psi^\star C_\Psi f, f \rangle_{X,X'} = \langle C_\Psi f, C_\Psi f \rangle_{\ell^2} = \|C_\Psi f\|_{\ell^2}^2 \geq A_\Psi \cdot \|f\|_{X'}^2, \tag{18}$$

which implies that S_Ψ is one-to-one. By [55, IV.5.1], this also means that $S_\Psi^\star = S_\Psi$ has dense range. S_Ψ is also positive since

$$\|S_\Psi f\|_X = \sup_{\substack{\|g\|_{X'}=1 \\ g \in X'}} \langle S_\Psi f, g \rangle_{X,X'} \geq \left\langle S_\Psi f, \frac{f}{\|f\|_{X'}} \right\rangle_{X,X'} \geq A_\Psi \cdot \|f\|_{X'}.$$

Therefore it has closed range [34, Theorem XI.2.1]. Consequently, S_Ψ is onto and bijective with

$$A_\Psi \|f\|_{X'} \leq \|S_\Psi f\|_X \leq B_\Psi \|f\|_{X'}.$$

Thus, S_Ψ^{-1} is also self-adjoint, and

$$\frac{1}{B_\Psi} \|g\|_X \leq \left\| S_\Psi^{-1} g \right\|_{X'} \leq \frac{1}{A_\Psi} \|g\|_X. \tag{19}$$

Theorem 4.1 *The sequence* $\widetilde{\Psi} = \left(\tilde{\psi}_k\right)_{k \in K} := \left(S_\Psi^{-1}\psi_k\right)_{k \in K} \subseteq X'$ *is a Stevenson Banach frame for* X *with bounds* $\frac{1}{B_\Psi}$ *and* $\frac{1}{A_\Psi}$. *The range of its analysis operator coincides with the one of the primal frame, i.e.* $\mathrm{ran}\left(C_\Psi\right) = \mathrm{ran}\left(C_{\widetilde{\Psi}}\right)$. *The related operators are* $C_{\widetilde{\Psi}} = C_\Psi S_\Psi^{-1}$, $D_{\widetilde{\Psi}} = S_\Psi^{-1} D_\Psi$ *and* $S_{\widetilde{\Psi}} = S_\Psi^{-1}$. *For* $f \in X$ *and* $g \in X'$, *we have the reconstructions*

$$f = \sum_{k \in K} \left\langle f, \tilde{\psi}_k\right\rangle_{X,X'} \psi_k \quad and \quad g = \sum_{k \in K} \langle g, \psi_k\rangle_{X',X} \tilde{\psi}_k.$$

Proof: It obviously holds $S_\Psi^{-1}\psi_k \in X'$. Moreover, we have on the one hand

$$\sum_{k \in K} \left|\left\langle f, \tilde{\psi}_k\right\rangle_{X,X'}\right|^2 = \sum_{k \in K} \left|\left\langle f, S_\Psi^{-1}\psi_k\right\rangle_{X,X'}\right|^2 = \sum_{k \in K} \left|\left\langle S_\Psi^{-1}f, \psi_k\right\rangle_{X',X}\right|^2$$

$$\leq B_\Psi \left\|S_\Psi^{-1}f\right\|_{X'}^2 \leq \frac{B_\Psi}{A_\Psi^2} \|f\|_X^2$$

and on the other hand

$$\sum_{k \in K} \left|\left\langle f, \tilde{\psi}_k\right\rangle_{X,X'}\right|^2 \geq A_\Psi \left\|S_\Psi^{-1}f\right\|_{X'}^2 \geq \frac{A_\Psi}{B_\Psi^2} \|f\|_{X'}^2.$$

Hence, $\widetilde{\Psi}$ is an X-frame. By employing the invertibility of S_Ψ for $g = S_\Psi^{-1}f$, we get

$$\left\langle f, S_\Psi^{-1}\psi_k\right\rangle_{X,X'} = \left\langle S_\Psi g, S_\Psi^{-1}\psi_k\right\rangle_{X,X'} = \left\langle g, S_\Psi S_\Psi^{-1}\psi_k\right\rangle_{X',X} = \langle g, \psi_k\rangle_{X',X}.$$

This implies $\mathrm{ran}\left(C_\Psi\right) = \mathrm{ran}\left(C_{\widetilde{\Psi}}\right)$, where $C_{\widetilde{\Psi}} = C_\Psi S_\Psi^{-1}$ and $D_{\widetilde{\Psi}} = S_\Psi^{-1} D_\Psi$. Furthermore, $S_{\widetilde{\Psi}} = S_\Psi^{-1} : X \to X'$, because it holds

$$S_{\widetilde{\Psi}} f = \sum_k \left\langle f, S_\Psi^{-1}\psi_k\right\rangle_{X,X'} S_\Psi^{-1}\psi_k$$

$$= S_\Psi^{-1} \sum_k \left\langle S_\Psi^{-1}f, \psi_k\right\rangle_{X,X'} \psi_k = S_\Psi^{-1} S_\Psi S_\Psi^{-1} g = S_\Psi^{-1} f$$

for all $f \in X$.

Finally, we have the reconstructions

$$f = D_\Psi C_\Psi S_\Psi^{-1} f = D_\Psi C_{\widetilde{\Psi}} f = \sum_{k \in K} \left\langle f, \tilde{\psi}_k\right\rangle_{X,X'} \psi_k$$

for all $f \in X$ and

$$g = S_\Psi^{-1} D_\Psi C_\Psi = D_{\widetilde{\Psi}} C_\Psi g = \sum_{k \in K} \langle h, \psi_k\rangle_{X',X} \tilde{\psi}_k$$

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for all $g \in X'$.

As
$$\left\langle S_\Psi^{-1} x, x \right\rangle_{X',X} \leq \left\| S_\Psi^{-1} x \right\|_{X'} \|x\|_X \leq \frac{1}{A_\Psi} \|x\|_X^2 \,,$$

we have the sharper upper bound. On the other hand, since $\left\langle S_\Psi^{-1} \cdot, \cdot \right\rangle$ defines a positive sesqui-linear form, the Cauchy-Schwarz inequality implies

$$\left| \left\langle S_\Psi^{-1} x, y \right\rangle_{X',X} \right|^2 \leq \left\langle S_\Psi^{-1} x, x \right\rangle_{X',X} \left\langle S_\Psi^{-1} y, y \right\rangle_{X',X} .$$

Thus, with $x = S_\Psi u$, there holds

$$\left| \langle u, y \rangle_{X',X} \right|^2 \leq \langle u, S_\Psi u \rangle_{X',X} \left\langle S_\Psi^{-1} y, y \right\rangle_{X',X}$$

and consequently

$$\|y\|_X^2 = \sup_{\substack{\|u\|_{X'}=1 \\ u \in X'}} \left| \langle u, y \rangle_{X',X} \right|^2 \leq B_\Psi \left\langle S_\Psi^{-1} y, y \right\rangle_{X',X} .$$

So, the sharper bounds $\frac{1}{B_\Psi}$ and $\frac{1}{A_\Psi}$ follow. $\qquad\square$

The fact that $\operatorname{ran}(C_\Psi) = \operatorname{ran}\left(C_{\widetilde{\Psi}}\right)$ is very different to the Gelfand frame setting, where the ranges $\operatorname{ran}\left(C_{\Psi|_{X'}}\right) \neq \operatorname{ran}\left(C_{\widetilde{\Psi}|_X}\right)$ even live in different sequence spaces.

Theorem 4.2 *The cross-Gramian matrix $G_{\Psi,\widetilde{\Psi}} = C_\Psi D_{\widetilde{\Psi}}$ is the orthogonal projection on $\operatorname{ran}(C_\Psi)$ and coincides with $G_{\widetilde{\Psi},\Psi}$.*

Proof: We have that the cross-Gramian matrix of a frame and its dual is a projection:

$$\left(G_{\Psi,\widetilde{\Psi}}\right)^2 = C_\Psi D_{\widetilde{\Psi}} C_\Psi D_{\widetilde{\Psi}} = C_\Psi D_{\widetilde{\Psi}}. = G_{\Psi,\widetilde{\Psi}}.$$

Next, it holds
$$G_{\Psi,\widetilde{\Psi}}^\star = \left(C_\Psi D_{\widetilde{\Psi}}\right)^\star = C_{\widetilde{\Psi}} D_\Psi = G_{\widetilde{\Psi},\Psi}.$$

In addition, since

$$\left(G_{\Psi,\widetilde{\Psi}}\right)_{k,l} = \left\langle \tilde{\psi}_l, \psi_k \right\rangle_{X',X} = \left\langle S_\Psi^{-1} \psi_l, \psi_k \right\rangle_{X',X} = \left\langle \psi_l, S_\Psi^{-1} \psi_k \right\rangle_{X,X'} = \left(G_{\widetilde{\Psi},\Psi}\right)_{k,l},$$

we conclude $G_{\Psi,\widetilde{\Psi}} = G_{\widetilde{\Psi},\Psi}$. Thus, $G_{\Psi,\widetilde{\Psi}}$ is self-adjoint. $\qquad\square$

Theorem 4.3 *The collection Ψ is a Stevenson Banach frame for X with bounds A_Ψ and B_Ψ if and only if*

$$\frac{1}{B_\Psi} \|f\|_X \leq \inf_{d \in \ell^2, \, D_\Psi d = f} \|d\|_{\ell^2} \leq \frac{1}{A_\Psi} \|f\|_X.$$

In particular, for any $f \in X$ with $f = \sum_{k \in K} d_k \psi_k$ and $d = (d_k) \in \ell^2$, we have $\|d\|_{\ell^2} \geq \|C_{\widetilde{\Psi}} f\|_{\ell^2}$.

Proof: Given

$$f = \sum_{k \in K} d_k \psi_k \in X,$$

we have the representation

$$f = \sum_{k \in K} \left\langle f, \tilde{\psi}_k \right\rangle_{X,X'} \psi_k.$$

Hence,

$$\left(d_k - \left\langle f, \tilde{\psi}_k \right\rangle_{X,X'} \right) \in \ker(D_\Psi).$$

By (16) and Theorem 4.1, there follows $\|d\|_{\ell^2} \geq \|C_{\widetilde{\Psi}} f\|_{\ell^2}$. $\qquad\square$

Consequently, a frame is a Riesz basis for X if and only if D_Ψ is one-to-one.

4.2 Is X' a Hilbert Space?

Set $\langle u, v \rangle_{X'_{\mathcal{H}}} := \langle u, S_\Psi v \rangle_{X',X}$. This is, trivially, a symmetric and positive bilinear form by above and, therefore, an inner product on X'. Hence, X' is a pre-Hilbert space with this inner product. By (17) and (18), the corresponding norm is equivalent to the original one as $\sqrt{A_\Psi} \|f\|_{X'} \leq \|f\|_{X'_{\mathcal{H}}} \leq \sqrt{B_\Psi} \|f\|_{X'}$. Thus, $(X', \langle \cdot, \cdot \rangle)$ is a Hilbert space.

Note that, in particular for numerics, it is sometimes not enough to consider equivalent norms. While well-posed problems stay well-posed for equivalent norms, the becomes important for concrete implementations, as things like condition numbers, constants in convergence rates, etc. are considered.

From a frame theory perspective, switching to an equivalent norm can destroy or create tightness, in particular, the switch from one norm to the other changes the frame bound ratio $\frac{B_\Psi}{A_\Psi}$. We refer the reader to e.g. weighted and controlled frames [7], which are under very mild conditions equivalent to classical Hilbert frames. Nonetheless, they have applications for example in

the implementation of wavelets on the sphere [15, 42], and nowadays become important for the scaling of frames [22, 45]. As a trivial example, look at $\Psi := \{e_1, e_1, e_2, e_2, e_3, e_3, \dots\}$, where $E = \{e_i\}_{i \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$. Then, Ψ is a tight frame with $A_\Psi = 2$. Looking at the reweighted version $\Phi := \{2e_1, 2e_1, e_2/2, e_2/2, 2e_3, 2e_3, \dots\}$, we loose tightness, since this frame has bounds $A = 1$ and $B = 4$. Note that here there exists an invertible bounded operator that maps the single element from Ψ into Φ.

Also note that, if it does not make sense to assume that $X \subset X'$, then Ψ cannot be a Hilbert space frame per se. This can only be true for $\Psi' := (\psi'_k) = (I\psi_k)$, where I is an isomorphism from X to X', for example, choosing $I = S_\Psi^{-1}$. In this case, the frame bounds are preserved, but the roles of primal and dual frames interchange.

This especially means that, if the frame bound ratio is important, distinguishing ℓ^2-Banach frames from Hilbert frames is necessary, especially if concrete examples for X and X' are used, where an identification is not possible, i.e. $X \neq X'$. As such, Definition 4.1 is, of course, equivalent to the standard frame definition for Hilbert spaces, but *the frame bound ratio changes*.

We like to remark that, by using the dual frame, one can also conclude that X itself is a Hilbert space.

4.3 Matrix Representation

Let us also revisit the statements about the matrix representation of operators [40, 52]. To this end, let Ψ be Stevenson Banach frame for X'.

Let us now consider an operator $O : X \to X'$ and define

$$\left(\mathcal{M}^{(\Psi)}(O)\right)_{m,n} = \langle O\psi_n, \psi_m \rangle_{X',X}.$$

Then, $\mathcal{M}^{(\Psi)}(O) = C_\Psi O D_\Psi$, which implies

$$\left\|\mathcal{M}^{(\Psi)}(O)\right\|_{\ell^2 \to \ell^2} \leq B_\Psi \|O\|_{X \to X'}.$$

(As in Section 3.6, we could consider different sequences, and the arguments would still work, but following the argument in the introduction and for easy reading we will not.)

For an invertible operator O, we have

$$\mathcal{M}^{(\widetilde{\Psi})}(O^{-1})\mathcal{M}^{(\Psi)}(O) = C_{\widetilde{\Psi}} O^{-1} D_{\widetilde{\Psi}} C_\Psi O D_\Psi = G_{\widetilde{\Psi},\Psi} = \pi_{\mathrm{ran}(C_\Psi)}.$$

(For the analogue result in the Hilbert frame case, see [11, 12].) Equivalently,

$$\mathcal{M}^{(\Psi)}(O)\mathcal{M}^{(\widetilde{\Psi})}(O^{-1}) = G_{\widetilde{\Psi},\Psi}.$$

Therefore, $\mathcal{M}^{(\Psi)}(O)_{|_{\mathrm{ran}(C_\Psi)}}$ is boundedly invertible, as

$$\left\|\mathcal{M}^{(\Psi)}(O)\right\|_{\mathrm{ran}(C_\Psi)\to\mathrm{ran}(C_\Psi)} \geq A_\Psi \left\|O^{-1}\right\|_{X'\to X}^{-1}.$$

Furthermore,

$$\ker\left(\mathcal{M}^{(\widetilde{\Psi})}(O)\right) = \ker\left(D_\Psi\right).$$

If O is symmetric, then $\mathcal{M}^{(\Psi)}(O)$ is symmetric. If O is non-negative, so is $\mathcal{M}^{(\Psi)}(O)$. In particular, we have now have settled all statements in [40, 52].

Acknowledgment

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